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STOCHASTIC INTEGRAL OF SKOROHOD TYPE WITH RESPECT TO RANDOM MEASURES

Generalization of Skorohod integral for the case of arbitrary random measures is constructed. Conditions of existence are established.

INTRODUCTION

In the present paper we construct an integral of a random function with respect to a random measure.

Random measures are defined in Section 3 as operators acting from a space of square integrable functions to a space of square integrable random variables. Often an operator will be bounded, but we use the weaker notion of regular operator defined in Section 2. Such definition allows us to consider Wiener integral as integral w.r.t. certain random measure. Another important example of random measure is a random measure χ on a Wiener space (μ, H, W) with a μ -distributed random element w . χ is defined by $\chi(\omega) = \delta_{w(\omega)}$ for $\omega \in \Omega$.

The difficulty of the problem of defining an integral of a random function with respect to a random measure can be illustrated by Wiener integral. There are two extensions of that for random functions, namely, Stratonovich and Skorohod ones. Stratonovich integral behaves like an integral 'for every ω ' and Skorohod integral is 'nice' from the functional analytic point of view — it is a closed operator. Possible generalizations of the notion of Stratonovich integral where considered in [1], [2]. In this paper we construct a generalization of Skorohod integral for the case of arbitrary random measures.

Our definition of integral will be based on the rule

$$\left(\int_X f(x) \nu(dx) \right) (w) = \left(\int_X f_{w_1}(x) \nu_{w_2}(dx) \right)_{w_1, w_2 = w}$$

which leads to the problem of restriction of a function defined on $W \times W$ to the diagonal. We solve it by constructing measure $\chi \otimes \chi$ (Section 4). Then needed restriction can be defined as integral w.r.t. $\chi \otimes \chi$. The measure is constructed so that its integration operator $\mathcal{N}$ commutes with Skorohod integral. We obtain direct formula for this operator which gives us sufficient conditions of existence of the integral (Section 6). In examples (Section 7) we show how this formula can be used to calculate the integral.

The obtained integral is extension of Skorohod integral. Also, in partial cases, the integral coincides with the action of measure as a random map on a function as a random element, which was defined in [4].

1. PRELIMINARIES

1.1 Tensor products. Let E_1, E_2 be linear spaces. $E_1 \otimes E_2$ will denote the algebraic tensor product. If, moreover, E_1 and E_2 are separable Hilbert spaces, the same sign will denote the closure with respect to the Hilbert-Schmidt norm. $E_1 \otimes_s E_2$ is the symmetric part of $E_1 \otimes E_2$.

When we consider a measurable space with a measure we will always assume that the σ -field is countably generated with respect to the measure. Put $E_1 = L^2(X_1, \mu_1)$ for some measurable space X_1 with a finite measure μ_1 . Then the tensor product $E_1 \otimes E_2$ can be identified with the space $L^2(X_1, \mu_1; E_2)$ of measurable maps from X_1 to E_2 having strong second moment. If $E_2 = L^2(X_2, \mu_2)$ then $E_1 \otimes E_2$ can also be identified with the space $L^2(X_1 \times X_2, \mu_1 \otimes \mu_2)$.

For linear operators $\phi_1: E_1 \rightarrow \tilde{E}_1, \phi_2: E_2 \rightarrow \tilde{E}_2, \phi_1 \otimes \phi_2$ will denote the operator tensor product acting from $E_1 \otimes E_2$ to $\tilde{E}_1 \otimes \tilde{E}_2$. If $E_1, E_2, \tilde{E}_1, \tilde{E}_2$ are separable Hilbert spaces and ϕ_1, ϕ_2 are closed and densely defined, then $\phi_1 \otimes \phi_2$ and $\phi_1 \otimes I + I \otimes \phi_2$ are closable and densely defined (see [7]) and we denote their closures again by $\phi_1 \otimes \phi_2$ and $\phi_1 \otimes I + I \otimes \phi_2$ respectively.

If $\phi: L^2(X_1, \mu_1) \rightarrow L^2(X_2, \mu_2)$ is a closed operator, then sometimes for $f \in L^2(X_1 \times X_2, \mu_1 \otimes \mu_2), f \in \mathcal{D}(\phi \otimes I)$ we will write $g(x, y) = \phi_x f(x, y)$ instead of $g = (\phi \otimes I)f$.

1.2 Necessary notions from stochastic analysis. Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a generalized Gaussian random element ξ in a real separable Hilbert space H such that ξ has zero mean, identity covariance and $\mathcal{F} = \sigma(\xi)$. Let H be densely imbedded by an operator i into a separable Banach space W such that $i(\xi) = w$ is a strong random element in W . Let μ be the distribution of w . The triple (μ, H, W) is an abstract Wiener space and we can use notions of Malliavin calculus (see [3], section 5.8). There is a natural correspondence between functions on W (precisely, classes of equivalence with respect to μ) and random variables on Ω .

Let us denote $L^2 = L^2(W, \mu)$. There is Ito-Wiener decomposition

$$L^2 = \bigoplus_{n=0}^{\infty} I_n(H^{\otimes n}),$$

where by $\bigoplus$ we mean the orthogonal sum of subspaces, I_n denotes multiple Wiener integral, $\frac{I_n}{\sqrt{n!}}$ preserves norm. Let $S \subset L^2$ be the dense linear subspace of polynomials, i.e. functions of the form

$$f = F((\xi, h_1), (\xi, h_2), \dots, (\xi, h_n))$$

for a polynomial $F: \mathbb{R}^n \rightarrow \mathbb{R}$ and $h_i \in H, i = 1, 2, \dots, n$.

Now the derivation operator ∇ acting from S to $S \otimes H$ can be defined by

$$\nabla F((\xi, h_1), (\xi, h_2), \dots, (\xi, h_n)) = \sum_{i=1}^n \nabla_i F((\xi, h_1), (\xi, h_2), \dots, (\xi, h_n)) \otimes h_i,$$

where ∇_i is the usual derivative w.r.t. i -th argument. Operator ∇ acting from L^2 to $L^2 \otimes H$ is closable, we denote its closure again by ∇ . Put $\delta = \nabla^*$ — the Skorohod integral and $\mathcal{L} = \delta \circ \nabla$ — the Ornstein-Uhlenbeck operator. The symbol $\circ$ denotes composition of operators.

For $h \in H$ and $1 \leq p < 2$ the bounded shift operator $T_h: L^2 \rightarrow L^p(W, \mu)$ can be defined as $(T_h f)(\cdot) = f(\cdot + h)$ a.s.

For a bounded operator A in H define closed operator $\Gamma(A): L^2 \rightarrow L^2$ by $\Gamma(A)I_n(f_n) = I_n(A^{\otimes n} f_n)$ for $f_n \in H^{\otimes n}$ (here $A^{\otimes 0} = I$). Clearly, if A is contracting then so is $\Gamma(A)$,

and for any operator B in H , $\Gamma(AB) = \Gamma(A)\Gamma(B)$. Operator $\mathcal{L}$ is self-adjoint, and its spectrum is $\mathbb{Z}_+ = \{0, 1, \dots\}$. For $\lambda \in \mathbb{R}$, $\lambda^{\mathcal{L}}$ is defined and coincides with $\Gamma(\lambda)$.

2. REGULAR OPERATORS

For any $\phi: \mathbb{Z}_+ \rightarrow \mathbb{R}$, $\phi(\mathcal{L})$ is a closed self-adjoint operator. The set of all such operators will be denoted by $F(\mathcal{L})$.

Definition 2.1. Let E be a Hilbert space. A linear operator $A: E \rightarrow L^2$ is regular iff

- (1) A is densely defined,
- (2) A admits a representation $A = FA'$, where $A': E \rightarrow L^2$ is bounded and $F \in F(\mathcal{L})$.

Clearly, every regular operator is closed and every bounded operator is regular. The next proposition allows us to define regular operators only on dense subsets.

Proposition 2.1. Let A_1, A_2 be regular operators from E to L^2 . If $A_1 = A_2$ on a dense subset then $A_1 = A_2$ as operators. If a densely defined operator has a regular extension then this extension is unique.

Proof can be done via multiplying both operators from the left by appropriate $\psi(\mathcal{L})$.

Example 2.1. Suppose, $A: E \rightarrow L^p(W, \mu)$ is a bounded operator for some $1 < p < 2$ and $Ax \in L^2$ for all x from some dense subset of E . Then the operator A is regular from E to L^2 .

Proof. We have to check only the second condition. By the theorem of hypercontractivity of Γ (see [3], section 5.8) there exists $\lambda \in (0; 1)$ such that $\Gamma(\lambda)$ acts continuously from $L^p(W, \mu)$ to L^2 . Thus obtain $A = (\Gamma(\lambda^{-1}))(\Gamma(\lambda)A)$ is the required representation. $\square$

3. RANDOM MEASURES AND RANDOM FUNCTIONS

Let X be a measurable space with some fixed finite measure μ_X .

Definition 3.1. We say that f is a random function on X if f belongs to $L^2(X \times W, \mu_X \otimes \mu)$.

Definition 3.2. We say that random measure ν on X is given if some regular operator $Int_\nu: L^2(X) \rightarrow L^2$ is put in correspondence to ν . Int_ν will be called the integration operator corresponding to ν and we put by definition for $f \in \mathcal{D}(Int_\nu)$

$$\int_X f(x) \nu(dx) = Int_\nu(f).$$

Example 3.1. Let X be $[0; 1]$ with Lebesgue measure, $H = L^2[0; 1]$. Then $w(t) = (\xi, 1_{[0; t]})$ is a standard Wiener process. Define a random measure $\dot{w}$ by

$$Int_{\dot{w}}(f) = (\xi, f) = \int_{[0; 1]} f(t) dw(t), \quad f \in L^2[0; 1].$$

It can be easily seen that $\dot{w}$ being defined in such way corresponds to the classical random measure with independent increments on $[0; 1]$, generated by Wiener process.

Example 3.2. Let X be W with the measure μ . We define a measure χ on X by $Int_\chi = I$ — the identity operator. The measure χ intuitively can be viewed as a one point measure, concentrated in the random point w . We will heavily use the fact that any random variable can be represented as an integral of a deterministic function with respect to the measure χ .

$$f(w) = \int_W f(x) \chi(dx), \quad f \in L^2.$$

Example 3.3. Suppose $(X, \mathcal{B})$ is a measurable space. Let $\nu: \Omega \times \mathcal{B} \rightarrow \mathbb{R}$ be such that

- (1) $\nu(\cdot, \omega)$ is a probability measure for all $\omega \in \Omega$,
- (2) $\nu(\Delta, \cdot)$ is a random variable for all $\Delta \in \mathcal{B}$.

The integration operator can be constructed in the following way. If $f = 1_\Delta$ then put

$$Int_\nu(f) = \nu(\Delta),$$

then extend by linearity for simple functions. Then for simple f

$$\mathbf{E}(Int_\nu(f))^2 = \mathbf{E} \left(\int_X f(x) \nu(dx) \right)^2 \leq \mathbf{E} \int_X f(x)^2 \nu(dx) = \int_X f(x)^2 \lambda(dx),$$

where $\lambda(\cdot) = \mathbf{E}\nu(\cdot)$, is the structure measure of ν . Therefore the integration operator can be uniquely extended to a contracting operator Int_ν from $L^2(X, \lambda)$ to L^2 and ν can be regarded as a random measure in the sense of Definition 3.2.

4. INTEGRATION OF RANDOM FUNCTIONS

Now we define a measure $\chi \otimes \chi$ on $W \times W$ corresponding to the multiple integral

$$\int_W \int_W f(x, y) \chi(dx) \chi(dy).$$

First we define $Int_{\chi \otimes \chi}$ for polynomials. Then we use regular extension to define it outside $S \otimes S$.

Definition 4.1. For $f = I_n(f_1) \otimes I_m(f_2) \in S \otimes S$ where $f_1 \in H^{\otimes n}$, $f_2 \in H^{\otimes m}$ put

$$(4.1) \quad Int_{\chi \otimes \chi}(f) = I_{n+m}(f_1 \otimes_s f_2),$$

then obtain for any $f \in S \otimes S$ by linearity. Then take regular extension (the existence of that will be proved in Theorem 4.1). For simplicity of notation denote $\mathcal{N} = Int_{\chi \otimes \chi}$.

Remark 4.1. It can be easily checked that this definition is the only possible one, satisfying:

$$\begin{aligned} Int_{\chi \otimes \chi}(1 \otimes f) &= f \quad \text{for } f \in S; \\ Int_{\chi \otimes \chi}(\delta \otimes I)f &= \delta(I \otimes Int_{\chi \otimes \chi})f \quad \text{for } f \in H \otimes S \otimes S. \end{aligned}$$

Remark 4.2. An expression of type (4.1) was used in [4] in order to define an actions of random maps on random elements. Also $I_{n+m}(f_1 \otimes_s f_2)$ is known as Wick product of $I_n(f_1)$ and $I_m(f_2)$ (see [5]).

Definition 4.2. A random function $f \in L^2 \otimes L^2(X)$ is called integrable w.r.t. ν if $f \in \mathcal{D}(\mathcal{N} \circ (Int_\nu \otimes I))$. If f is integrable then

$$\int_X f(x) \nu(dx) = \mathcal{N}(Int_\nu \otimes I)f.$$

Now we discover some properties of the operator $\mathcal{N}$. Here $\mathbf{E}: L^2 \rightarrow \mathbb{R}$ is the mathematical expectation, $\nabla_h(\cdot) = (\nabla(\cdot), h)$ for $h \in H$.

Proposition 4.1. Let $\mathcal{N}$ be the operator, defined in Definition 4.1. Then

- (1) $\mathbf{E} \circ \mathcal{N} = \mathbf{E} \otimes \mathbf{E}$;
- (2) $\nabla_h \circ \mathcal{N} = \mathcal{N} \circ (\nabla_h \otimes I + I \otimes \nabla_h)$ for $h \in H$;
- (3) $T_h \circ \mathcal{N} = \mathcal{N} \circ (T_h \otimes T_h)$ for $h \in H$.

Proof. (1) and (2) can be easily checked for elements of type $I_n(f_1) \otimes I_m(f_2)$ using identities for $f \in H^{\otimes k}$, $k \in \mathbb{Z}_+$:

$$\mathbf{E}I_k(f) = \begin{cases} 0, & \text{if } k > 0 \\ f, & \text{if } k = 0 \end{cases}, \quad \nabla_h I_k(f) = \begin{cases} I_{k-1}((f, h)), & \text{if } k > 0 \\ 0, & \text{if } k = 0 \end{cases}.$$

For proving (3) we can use formula

$$T_h = \exp(\nabla_h),$$

where the righthand side is a power series, which is finite for elements of S , and now (3) easily follows from (2) and algebraic identity

$$\exp(\nabla_h \otimes I + I \otimes \nabla_h) = \exp(\nabla_h) \otimes \exp(\nabla_h).$$

□

Theorem 4.1. *The Definition 4.1 is correct, moreover*

$$(4.2) \quad \mathcal{N} = \Gamma(\sqrt{2}) \circ \mathcal{N}_0,$$

where

$$(\mathcal{N}_0 f)(x) = \int_W f\left(\frac{x+\alpha}{\sqrt{2}}, \frac{x-\alpha}{\sqrt{2}}\right) \mu(d\alpha)$$

is a bounded operator

Proof. Define a mapping r from $W \times W$ to $W \times W$ by

$$(x, \alpha) \mapsto \left(\frac{x+\alpha}{\sqrt{2}}, \frac{x-\alpha}{\sqrt{2}}\right).$$

Then r preserves measure $\mu \otimes \mu$, so the linear operator $r^*: L^2 \otimes L^2 \rightarrow L^2 \otimes L^2$ given by $(r^* f)(x, \alpha) = f(r(x, \alpha))$ is well defined and preserves norm. It follows that the operator $\Gamma(\sqrt{2}) \circ \mathcal{N}_0$ is regular and it is left to prove the identity for $f \in S \otimes S$.

Use invertible transform $(\mathcal{G}f)(h) = \mathbf{E}T_h f$ acting from L^2 to the space of real analytical functions on H .

$$\begin{aligned} (\mathcal{G}\mathcal{N}f)(h) &= \mathbf{E}T_h \mathcal{N}f = (\mathbf{E} \otimes \mathbf{E})(T_h \otimes T_h)f \\ &= \int_{W \times W} f(x+h, y+h) \mu(dx) \mu(dy) \\ &= \int_{W \times W} f\left(\frac{x+\alpha}{\sqrt{2}} + h, \frac{x-\alpha}{\sqrt{2}} + h\right) \mu(dx) \mu(d\alpha) \\ &= \mathbf{E}T_{\sqrt{2}h} \mathcal{N}_0 f = (\mathcal{G}\mathcal{N}_0 f)(\sqrt{2}h) = (\mathcal{G}\Gamma(\sqrt{2})\mathcal{N}_0 f)(h). \end{aligned}$$

□

The formula (4.2) becomes more natural if we note that in order to obtain a restriction of a function on $\mathbb{R} \times \mathbb{R}$ to the diagonal we can apply rotation by $-\pi/4$, then substitute y by 0, then contract by $\sqrt{2}$.

Then the following easily follows.

Corollary 4.1. *A random function $f \in L^2 \otimes L^2(X)$ is integrable w.r.t. ν iff $f \in \mathcal{D}(I \otimes \text{Int}_\nu)$ and*

$$\int_{X \times W} f\left(x, \frac{w+\alpha}{\sqrt{2}}\right) \nu\left(dx, \frac{w-\alpha}{\sqrt{2}}\right) \mu(d\alpha) \in \mathcal{D}(\Gamma(\sqrt{2})).$$

If f is integrable then

$$(4.3) \quad \int_X f(x) \nu(dx) = \Gamma(\sqrt{2})_w \int_{X \times W} f\left(x, \frac{w+\alpha}{\sqrt{2}}\right) \nu\left(dx, \frac{w-\alpha}{\sqrt{2}}\right) \mu(d\alpha).$$

5. PROPERTIES OF THE OPERATOR $\mathcal{N}$

In this section we study useful properties of operator $\mathcal{N}$. Consider operators $\mathcal{L} \otimes I$ and $I \otimes \mathcal{L}$. Since they are commuting, for any $\phi: \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{R}$ the closed operator $\phi(\mathcal{L} \otimes I, I \otimes \mathcal{L}): L^2 \otimes L^2 \rightarrow L^2 \otimes L^2$ can be defined in a usual way with help of operator valued measures. Similarly to Definition 2.1 we can define regular operators from E to $L^2 \otimes L^2$, where E is a Hilbert space. It can be checked that the proof of the Theorem 2.1 remains valid for this case.

Proposition 5.1. *If $\phi: \mathbb{Z}_+ \rightarrow \mathbb{R}$ is bounded then*

$$\mathcal{N} \circ \phi(\mathcal{L} \otimes I + I \otimes \mathcal{L}) \supset \phi(\mathcal{L}) \circ \mathcal{N}.$$

Proof. First, note that the operator in the lefthand side is regular since it is defined on polynomials and is composition of a bounded operator and a regular operator. Thus, it is enough to prove that operators agree on polynomials and that the operator in the righthand side has regular extension.

If $f = I_n(f_1) \otimes I_m(f_2) \in S \otimes S$, where $f_1 \in H^{\otimes n}$, $f_2 \in H^{\otimes m}$ then

$$\mathcal{N} \phi(\mathcal{L} \otimes I + I \otimes \mathcal{L}) f = \phi(\mathcal{L}) \mathcal{N} f = \phi(n+m) I_{m+n}(f_1 \otimes_s f_2)$$

follows from the fact, that f is an eigenvector of operators $\mathcal{L} \otimes I$, $I \otimes \mathcal{L}$ with eigenvalues n , m respectively and $\mathcal{N} f$ is an eigenvector of operator $\mathcal{L}$ with eigenvalue $m+n$. Then by linearity obtain this result for all polynomials.

Now prove that $\phi(\mathcal{L}) \circ \mathcal{N}$ has regular extension:

$$\phi(\mathcal{L}) \circ \mathcal{N} = \phi(\mathcal{L}) \circ \Gamma(\sqrt{2}) \circ \mathcal{N}_0 \subset \Gamma(\sqrt{2}) \circ \phi(\mathcal{L}) \circ \mathcal{N}_0.$$

The last operator is regular. $\square$

Theorem 5.1. *If $A: E \rightarrow L^2 \otimes L^2$ is regular operator, $\mathcal{N} \circ A$ is densely defined, then $\mathcal{N} \circ A$ has regular extension.*

Proof. Let $A = F_A \circ A_0 = \phi(\mathcal{L} \otimes I, I \otimes \mathcal{L}) \circ A_0$, $\phi: \mathbb{Z}_+ \otimes \mathbb{Z}_+ \rightarrow \mathbb{R}$. There exists $\psi: \mathbb{Z}_+ \rightarrow \mathbb{R}$ such that $|\phi(n, m)| + 1 \leq \psi(n+m)$, for all $n, m \in \mathbb{Z}_+$. Then, $B = \psi^{-1}(\mathcal{L} \otimes I + I \otimes \mathcal{L})$ is bounded and B^{-1} is defined.

$$A = B^{-1} \circ B \circ F_A \circ A_0 \subset B^{-1} \circ F_A \circ B \circ A_0.$$

Operator $F_A \circ B \circ A_0$ is bounded, so $B^{-1} \circ F_A \circ B \circ A_0$ is regular hence equal to A .

It is easy to check that $\mathcal{N} \circ B \supset \psi(\mathcal{L}) \circ \mathcal{N}$ implies $\mathcal{N} \circ B^{-1} \subset \psi(\mathcal{L})^{-1} \circ \mathcal{N}$. So

$$\begin{aligned} \mathcal{N} \circ A &= \mathcal{N} \circ B^{-1} \circ F_A \circ B \circ A_0 \subset B^{-1} \circ \mathcal{N} \circ F_A \circ B \circ A_0 \\ &= \psi(\mathcal{L}) \circ \Gamma(\sqrt{2}) \circ \mathcal{N}_0 \circ F_A \circ B \circ A_0. \end{aligned}$$

$\mathcal{N}_0 \circ F_A \circ B \circ A_0$ is bounded. $\psi(\mathcal{L}) \circ \Gamma(\sqrt{2}) \subset \psi'(\mathcal{L})$, where $\psi': \mathbb{Z}_+ \rightarrow \mathbb{R}$ is given by $\psi'(n) = \psi(n)(\sqrt{2})^n$. Thus, the theorem is proved. $\square$

Recall that tensor product of closed operators is closure of algebraic tensor product.

Corollary 5.1. *If $A_1: E_1 \rightarrow L^2$, $A_2: E_2 \rightarrow L^2$ are regular operators and if operator $\mathcal{N} \circ (A_1 \otimes A_2)$ is densely defined, then it has regular extension.*

Proof. If $A_1 = F_1 \circ A'_1$ and $A_2 = F_2 \circ A'_2$ are representations from Definition 2.1 then

$$A_1 \otimes A_2 \subset (F_1 \otimes F_2) \circ (A'_1 \otimes A'_2),$$

since both operators are closed, they agree on $\mathcal{D}A_1 \otimes \mathcal{D}A_2$, and the operator in lefthand side is closure of it's restriction to $\mathcal{D}A_1 \otimes \mathcal{D}A_2$. The operator in righthand side is regular, which implies the statement. $\square$

6. EXISTENCE OF THE INTEGRAL

We give some sufficient conditions of a function to belong to the domain of $\mathcal{N}$, which can help to prove existence of stochastic integrals.

Theorem 6.1. *Suppose $f \in L^2 \otimes L^2$ is such that*

$$f = (\Gamma(\sqrt{2}/2) \otimes \Gamma(\sqrt{2}/2))g \quad \text{for some } g \in L^2 \otimes L^2.$$

Then $f \in \mathcal{D}(\mathcal{N})$ and $\mathcal{N}f = \mathcal{N}_0g$.

Proof. Put $\phi(n) = 2^{-\frac{n}{2}}$ and apply Proposition 5.1:

$$\Gamma(\sqrt{2}/2) \circ \mathcal{N} \subset \mathcal{N} \circ (\Gamma(\sqrt{2}/2) \otimes \Gamma(\sqrt{2}/2)).$$

On the other hand:

$$\Gamma(\sqrt{2}/2) \circ \mathcal{N} = \Gamma(\sqrt{2}/2) \circ \Gamma(\sqrt{2})\mathcal{N}_0 \subset \mathcal{N}_0.$$

So

$$\mathcal{N} \circ (\Gamma(\sqrt{2}/2) \otimes \Gamma(\sqrt{2}/2)) = \mathcal{N}_0.$$

$\square$

Corollary 6.1. *Suppose a random measure ν is such that values of Int_ν belong to the domain of definition of $\Gamma(\sqrt{2})$ and let $\nu' = \Gamma(\sqrt{2})\nu$ be a random measure given by $\text{Int}_{\nu'} = \Gamma(\sqrt{2}) \circ \text{Int}_\nu$. Suppose a random function $f \in L^2(X) \otimes L^2$ is such that it belongs to the domain of definition of $\Gamma(\sqrt{2})_w$ and $f' = \Gamma(\sqrt{2})_w f$. Then the integral of f w.r.t. ν exists and*

$$\int_X f(x)\nu(dx) = \int_{X \times W} f' \left(x, \frac{w + \alpha}{\sqrt{2}} \right) \nu' \left(dx, \frac{w - \alpha}{\sqrt{2}} \right) \mu(d\alpha)$$

Theorem 6.2. *Suppose $f \in L^2 \otimes L^2$ is such that*

(1) *f has finite Ito-Wiener expansion w.r.t. the first argument, i.e.*

$$f = \sum_{n=0}^N (I_n \otimes I) f_n, \quad \text{where } f_n \in H^{\otimes_s n} \otimes L^2, \quad n = 0, 1, \dots, N.$$

(2) *f is N times stochastically differentiable w.r.t. second argument, i.e. $f \in \mathcal{D}(I \otimes \nabla^N)$.*

Then $f \in \mathcal{D}(\mathcal{N})$ and

$$(6.1) \quad \mathcal{N}f = \sum_{n=0}^N \delta^n f_n.$$

Proof. Clearly, for $n \in \mathbb{Z}_+$, δ^n is regular (see Remark 2.1). So is $\mathcal{N} \circ (I_n \otimes I)$. Moreover, for $g_1 \in H^{\otimes_s n}$, $g_2 \in H^{\otimes_s m}$, where $m \in \mathbb{Z}_+$ we have

$$\delta^n(g_1 \otimes I_m(g_2)) = I_{n+m}(g_1 \otimes_s g_2) = \mathcal{N}(I_n \otimes I)(g_1 \otimes I_m(g_2)),$$

which implies $\delta^n = \mathcal{N} \circ (I_n \otimes I)$. Since $f \in \mathcal{D}(I \otimes \nabla^n)$ so is f_n for $n = 0, 1, \dots, N$. It follows that the righthand side in (6.1) is defined, which completes the proof. $\square$

Corollary 6.2. *Suppose a random measure ν is such that*

$$\int_X f(x) \nu(dx) = \sum_{n=0}^N I_n(\nu_n f), \quad \text{for all } f \in L^2(X),$$

for some bounded operators $\nu_0, \nu_1, \dots, \nu_N$ acting from $L^2(X)$ to $H^{\otimes_s 0}, H^{\otimes_s 1}, \dots, H^{\otimes_s N}$ respectively. Suppose a random function f is N times stochastically differentiable. Then the integral of f w.r.t. ν exists and

$$\int_X f(x) \nu(dx) = \sum_{n=0}^N \delta^n(\nu_n f)$$

Remark 6.1. This shows that for random measure w the integral coincides with Skorohod integral.

Remark 6.2. In [4] action of a random operator on a random element was defined and similar theorem was obtained. This theorem shows that for a measure and a function satisfying conditions of the theorem, the integral coincides with the action of the integration operator on the random element in the space of functions.

Remark 6.3. Similar result can be proved for differentiable measures and functions with finite Ito-Wiener expansion.

Next we consider the case of random delta-measures

Let η be a random element in a separable Hilbert space E and f be a random map from E to $\mathbb{R}$ continuous in L^2 norm. In [4], the action of f on η was defined. Suppose λ is distribution of η and $f \in L^2(E, \lambda) \otimes L^2$. Then $f \circ \eta \in L^2 \otimes L^2$ ($\circ$ denotes simple composition of $f: E \times \Omega \rightarrow \mathbb{R}$ and $\eta: \Omega \rightarrow E$) has Ito-Wiener expansion

$$f \circ \eta = \sum_{n=0, m=0}^{\infty} (I_n \otimes I_m) f_{nm}, \quad \text{where } f_{nm} \in H^{\otimes_s n} \otimes H^{\otimes_s m}, n, m = 0, 1, \dots$$

If we omit localization procedure then $f(\eta)$ is defined as

$$(6.2) \quad f(\eta) = \sum_{n=0, m=0}^{\infty} I_{n+m}(\Lambda f_{nm}), \quad \text{where } \Lambda \text{ denotes the symmetrization,}$$

if the sum converges. On the other hand, η generates the random measure δ_η — one point measure concentrated in η . Then $(I \otimes \text{Int}_{\delta_\eta})f$ equals to $f \circ \eta$. Convergence of series in (6.2) and closedness of $\mathcal{N}$ imply $f \circ \eta \in \mathcal{D}(\mathcal{N})$. So the integral of f with respect to δ_η is defined and

$$f(\eta) = \int_E f(x) \delta_\eta(dx).$$

Note, that for $g: \mathbb{R} \rightarrow \mathbb{R}$ the $g(f(\eta))$ in general doesn't equal to $(g \circ f)(\eta)$, so this object is not an 'action' in the intuitive sense.

In [4] existence of the composition was proved in the case when f has finite Ito-Wiener expansion w.r.t. probability argument and is smooth w.r.t. space argument, η is smooth. We will prove the existence of integral in the case when f is smooth w.r.t probability argument and is k -linear form w.r.t. space argument, η has finite Ito-Wiener expansion. Suppose

$$\eta = \sum_{n=0}^N (I \otimes I_n) f_n, \quad \text{where } f_n \in E \otimes H^{\otimes_s n}, n = 0, 1, \dots, N.$$

Since on $I_n(H^{\otimes n})$, L^p norms are equivalent $1 < p < \infty$, we can deduce that $\|\eta\|$ has all moments finite. It follows that if B is a continuous k -linear form on H then its value on η is a random variable with all moments finite.

Theorem 6.3. *If B is a random element with strong second moment in the space of continuous k -linear forms on H , B is kN times stochastically differentiable, then the integral $\int_E B(x)\delta_\eta(dx)$ is defined.*

Proof. First,

$$\mathbf{E} \int_E B^2(x) \lambda(dx) \leq \mathbf{E} \|B\|^2 \mathbf{E} \|\eta\|^k,$$

so $B \in L^2(E, \lambda) \otimes L^2$. It is enough to prove that Int_{δ_η} maps deterministic k -linear forms to

$$\bigoplus_{i=0}^{kN} H^{\otimes i}.$$

Then we could apply Theorem 6.2 since $(\text{Int}_{\delta_\eta} \otimes I)f$ would have Ito-Wiener expansion w.r.t. first argument with forms of degree not exceeding kN and have kN derivatives w.r.t. second argument.

Since any k -linear form can be pointwise approximated by a sequence of finite dimensional k -linear forms with bounded norms, by the dominated convergence theorem deduce that it can be approximated in $L^2(E, \lambda)$ norm. So it is enough to prove the statement for forms of type $B = (x, e_1)(x, e_2) \dots (x, e_k)$, where $e_1, e_2, \dots, e_k \in E$. We have that $\nabla^{N+1}(\eta, e_i) = 0$ for $i = 0, 1, \dots, k$. Then, applying the chain rule it can be checked that $\nabla^{Nk+1}B$ is defined and equals to zero, which implies the required statement. $\square$

7. EXAMPLES

This section contains examples of calculating the integral with help of (4.2) and (4.3).

7.1 Multiple integral. Suppose that $f \in \mathcal{D}(\mathcal{N})$. It is clear that the double integral

$$\int_{W \times W} f(x, y) \chi(dx) \chi(dy)$$

is commutative, i.e.

$$\int_{W \times W} f(x, y) \chi(dx) \chi(dy) = \int_{W \times W} f(y, x) \chi(dx) \chi(dy)$$

In this example we consider the triple integral and prove that it is associative, precisely if $f \in L^2 \otimes L^2 \otimes L^2$, then

$$\int_{W \times W \times W} f(x, y, z) \chi(dx) \chi(dy) \chi(dz) = \int_{W \times W \times W} f(z, x, y) \chi(dx) \chi(dy) \chi(dz)$$

if all the integrals exist.

We will use directly the formula (4.2), the Mehler's formula for $\Gamma(1/2)$ (See [6]) and the commutation of $\mathcal{N}$ and Γ . For that we apply $\Gamma(1/2)$ to the integral.

$$\begin{aligned}
 \Gamma(1/2)_w \int_{W \times W \times W} f(x, y, z) \chi(dx) \chi(dy) \chi(dz) \\
 &= \Gamma(1/2)_w \int_W \left(\Gamma(1/2)_w \Gamma(1/2)_z \int_{W \times W} f(x, y, z) \chi(dx) \chi(dy) \right) \chi(dz) \\
 &= \Gamma(1/2)_w \int_{W^4} f\left(\frac{x+\alpha}{\sqrt{2}}, \frac{x-\alpha}{\sqrt{2}}, \frac{z+\beta}{\sqrt{2}}\right) \mu(d\alpha) \mu(d\beta) \chi(dx) \chi(dz) \\
 &= \int_{W^4} f\left(\frac{x+\gamma+\sqrt{2}\alpha}{2}, \frac{x+\gamma-\sqrt{2}\alpha}{2}, \frac{x-\gamma+\sqrt{2}\beta}{2}\right) \mu(d\alpha) \mu(d\beta) \mu(d\gamma) \chi(dx).
 \end{aligned}$$

Next consider the distribution of

$$(\alpha', \beta', \gamma') = \left(\frac{\gamma}{2} + \frac{\alpha}{\sqrt{2}}, \frac{\gamma}{2} - \frac{\alpha}{\sqrt{2}}, -\frac{\gamma}{2} + \frac{\beta}{\sqrt{2}} \right) \in W^3,$$

where α, β, γ are independent μ -distributed random elements in W . Check that the average

$$s = \frac{\alpha' + \beta' + \gamma'}{3} = \frac{\gamma}{6} + \frac{\beta}{3\sqrt{2}} \in W$$

and $(\alpha' - s, \beta' - s, \gamma' - s)$ in W^3 are independent:

$$\begin{aligned}
 \text{cov}(s, \alpha' - s) &= \text{cov}\left(\frac{\gamma}{6} + \frac{\beta}{3\sqrt{2}}, \frac{\gamma}{3} + \frac{\alpha}{\sqrt{2}} - \frac{\beta}{3\sqrt{2}}\right) = \frac{\text{cov}(\gamma, \gamma) - \text{cov}(\beta, \beta)}{18} = 0 \\
 \text{cov}(s, \beta' - s) &= \text{cov}\left(\frac{\gamma}{6} + \frac{\beta}{3\sqrt{2}}, \frac{\gamma}{3} - \frac{\alpha}{\sqrt{2}} - \frac{\beta}{3\sqrt{2}}\right) = \frac{\text{cov}(\gamma, \gamma) - \text{cov}(\beta, \beta)}{18} = 0 \\
 \text{cov}(s, \gamma' - s) &= \text{cov}\left(\frac{\gamma}{6} + \frac{\beta}{3\sqrt{2}}, -\frac{2\gamma}{3} + \frac{\sqrt{2}\beta}{3}\right) = \frac{\text{cov}(\beta, \beta) - \text{cov}(\gamma, \gamma)}{9} = 0
 \end{aligned}$$

Then, calculate the covariance of s .

$$\text{cov}(s, s) = \text{cov}\left(\frac{\gamma}{6} + \frac{\beta}{3\sqrt{2}}, \frac{\gamma}{6} + \frac{\beta}{3\sqrt{2}}\right) = \frac{\text{cov}(\gamma, \gamma)}{36} + \frac{\text{cov}(\beta, \beta)}{18} = \frac{\text{cov}(w, w)}{12}$$

Now, consider independent μ -distributed elements

$$\alpha_0 = \alpha, \quad \beta_0 = \frac{\sqrt{2}\gamma}{\sqrt{3}} - \frac{\beta}{\sqrt{3}},$$

$$\gamma_0 = \frac{\gamma}{\sqrt{3}} + \frac{\sqrt{2}\beta}{\sqrt{3}} = 2\sqrt{3}s.$$

Let us calculate

$$\begin{aligned}
 &\left(\frac{x}{2} + \alpha', \frac{x}{2} + \beta', \frac{x}{2} + \gamma' \right) = \\
 &\left(\frac{x}{2} + \frac{\alpha_0}{\sqrt{2}} + \frac{\beta_0}{\sqrt{6}} + \frac{\gamma_0}{2\sqrt{3}}, \frac{x}{2} - \frac{\alpha_0}{\sqrt{2}} + \frac{\beta_0}{\sqrt{6}} + \frac{\gamma_0}{2\sqrt{3}}, \frac{x}{2} - \frac{\sqrt{2}\beta_0}{\sqrt{3}} + \frac{\gamma_0}{2\sqrt{3}} \right) \\
 &= \left(\frac{1}{\sqrt{3}} \left(\frac{\sqrt{3}x}{2} + \frac{\gamma}{2} \right) + \frac{\alpha_0}{\sqrt{2}} + \frac{\beta_0}{\sqrt{6}}, \frac{1}{\sqrt{3}} \left(\frac{\sqrt{3}x}{2} + \frac{\gamma}{2} \right) - \frac{\alpha_0}{\sqrt{2}} + \frac{\beta_0}{\sqrt{6}}, \frac{1}{\sqrt{3}} \left(\frac{\sqrt{3}x}{2} + \frac{\gamma}{2} \right) - \frac{\sqrt{2}\beta_0}{\sqrt{3}} \right)
 \end{aligned}$$

So the considered integral equals to

$$\begin{aligned}
 &\Gamma\left(\frac{\sqrt{3}}{2}\right)_w \int_{W \times W \times W} f\left(\frac{x}{\sqrt{3}} + \frac{\alpha_0}{\sqrt{2}} + \frac{\beta_0}{\sqrt{6}}, \frac{x}{\sqrt{3}} - \frac{\alpha_0}{\sqrt{2}} + \frac{\beta_0}{\sqrt{6}}, \frac{x}{\sqrt{3}} - \frac{\sqrt{2}\beta_0}{\sqrt{3}}\right) \\
 &\times \mu(d\alpha_0) \mu(d\beta_0) \chi(dx) = \Gamma\left(\frac{\sqrt{3}}{2}\right)_w \int_{x+y+z=\sqrt{3}w} f(x, y, z) \mu^3(dx \times dy \times dz),
 \end{aligned}$$

as is easy to see. Here the last integral means the conditional expectation of f considered as a random variable on the probability space W^3 . The formula for the triple integral comes immediately.

$$\begin{aligned} & \int_{W \times W \times W} f(x, y, z) \chi(dx) \chi(dy) \chi(dz) \\ &= \Gamma(\sqrt{3})_w \int_{x+y+z=\sqrt{3}w} f(x, y, z) \mu^3(dx \times dy \times dz). \end{aligned}$$

Obviously it doesn't depend on the order of arguments.

7.2 Non-positivity. Next we will show a surprising fact that an integral of an a.s. nonnegative random function w.r.t. a random probability measure is not necessarily nonnegative a.s.

Let $X = H = W = \mathbb{R}$. Consider a probability measure m on $\mathbb{R}$ with density w.r.t. Lebesgue measure given by

$$\frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Suppose $w = \xi$ is a white noise and $m'(\cdot, w)$ be m shifted by w . m' is a random measure. Define a nonnegative random function f by

$$f(x, w) = (x - aw)^2.$$

Here $a \in \mathbb{R}$ is some constant. Then, by (4.3)

$$\int_{\mathbb{R}} f(x) m(dx) = \Gamma(\sqrt{2})_w \int_{\mathbb{R} \times \mathbb{R}} f\left(x, \frac{w+\alpha}{\sqrt{2}}\right) m'\left(dx, \frac{w-\alpha}{\sqrt{2}}\right) \mu(d\alpha).$$

Using the definition of f and m' obtain that it equals to

$$\begin{aligned} & \Gamma(\sqrt{2})_w \int_{\mathbb{R} \times \mathbb{R}} f\left(x + \frac{w-\alpha}{\sqrt{2}}, \frac{w+\alpha}{\sqrt{2}}\right) m(dx) \mu(d\alpha) \\ &= \Gamma(\sqrt{2})_w \int_{\mathbb{R} \times \mathbb{R}} \left(x + \frac{w-\alpha}{\sqrt{2}} - a \frac{w+\alpha}{\sqrt{2}}\right)^2 x^2 \mu(dx) \mu(d\alpha). \end{aligned}$$

Then, use properties of Gaussian measure:

$$\begin{aligned} &= \Gamma(\sqrt{2})_w \int_{\mathbb{R}} \left(3 + \left(\frac{w-\alpha}{\sqrt{2}} - a \frac{w+\alpha}{\sqrt{2}}\right)^2\right) \mu(d\alpha) \\ &= \Gamma(\sqrt{2})_w \left(3 + \frac{(a-1)^2}{2} w^2 + \frac{(a+1)^2}{2}\right) \\ &= \Gamma(\sqrt{2})_w \left(\frac{(a-1)^2}{2} (w^2 - 1) + a^2 + 4\right). \end{aligned}$$

Calculate $\Gamma(\sqrt{2})$ and obtain

$$= (a-1)^2 (w^2 - 1) + a^2 + 4 = 3 + 2a + (a+1)^2 w^2.$$

It easy to see that if $3 + 2a < 0$ then the last expression is not a.s. nonnegative.

7.3 Random maps. When a random measure is an image of a deterministic measure under a random map, the formula (4.3) simplifies.

Proposition 7.1. Suppose ν_0 is a probability measure on a measurable space X , F is a measurable mapping from $X \times \Omega$ to a measurable space Y . Let $\nu(\omega) = F^{-1}(\cdot, \omega)\nu_0$ for every $\omega \in \Omega$. Clearly, ν can be viewed as a random measure. Then for a random function f

$$\int_Y f(x)\nu(dx) = \Gamma(\sqrt{2})_w \int_{X \times W} f\left(F\left(x, \frac{w-\alpha}{\sqrt{2}}\right), \frac{w+\alpha}{\sqrt{2}}\right) \nu_0(dx)\mu(d\alpha).$$

Proof. For a deterministic function f

$$\text{Int}_\nu f = \int_X f(F(x, w))\nu_0(dx).$$

Then the statement follows from (4.3). $\square$

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