

Algebraic Topology

Lecture Notes

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1 Topological spaces and homotopy

Lecture 1, 1 October 2026

Throughout these notes, $I = [0, 1]$ denotes the unit interval.

1.1 The study of shapes

Topology studies shapes and continuous maps between them. Familiar examples include Euclidean space $\mathbb{R}^n$, the spheres

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\},$$

the torus (the surface of a doughnut), and a surface of genus two (a surface with two handles). We would like to say when two spaces have the same shape, and to find ways of proving that their shapes are different.

There are several possible meanings of “the same shape.” A homeomorphism is a continuous change of coordinates with a continuous inverse. Homotopy equivalence, introduced later in this lecture, allows more flexibility: for example, a solid ball will have the same homotopy type as a point. Algebraic topology studies spaces by associating algebraic objects to them. Such objects can provide *invariants*: quantities or structures that remain unchanged under the chosen notion of equivalence. Our first task is to make the topological language precise.

1.2 Topologies and metric spaces

In analysis, continuity is expressed using distances and ε - δ conditions. Topology extracts the part of that language which depends only on open sets.

Definition 1.1. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset$ and X belong to $\mathcal{T}$;
2. an arbitrary union of members of $\mathcal{T}$ belongs to $\mathcal{T}$;
3. a finite intersection of members of $\mathcal{T}$ belongs to $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is a *topological space*, and the members of $\mathcal{T}$ are its *open sets*. We usually write simply X when the topology is understood.

Arbitrary unions include infinite unions. Intersections, by contrast, are required to be open only when finitely many open sets are involved. For example, in the usual topology on $\mathbb{R}$,

$$\bigcap_{m=1}^{\infty} (-1/m, 1/m) = \{0\}$$

is not open.

Definition 1.2. A *metric* on a set X is a function $d : X \times X \rightarrow [0, \infty)$ such that, for all $x, y, z \in X$,

$$\begin{aligned} d(x, y) = 0 &\iff x = y, \\ d(x, y) &= d(y, x), \\ d(x, z) &\leq d(x, y) + d(y, z). \end{aligned}$$

A set equipped with a metric is a *metric space*. Its open ball with centre x and radius $r > 0$ is

$$B_d(x, r) = \{y \in X : d(x, y) < r\}.$$

A subset $U \subseteq X$ is *open in the metric topology* if, for every $x \in U$, there is an $r > 0$ such that $B_d(x, r) \subseteq U$.

Proposition 1.3. *The open subsets of a metric space form a topology.*

Proof. The empty set satisfies the condition vacuously, and X contains every ball. If $x \in \bigcup_{\alpha} U_{\alpha}$, choose α with $x \in U_{\alpha}$. A ball about x contained in U_{α} is also contained in the union. If $x \in U_1 \cap \cdots \cap U_k$, choose $r_i > 0$ with $B_d(x, r_i) \subseteq U_i$. The ball of radius $\min(r_1, \dots, r_k) > 0$ lies in the intersection. The empty intersection is X . $\square$

The balls themselves are open. If $y \in B_d(x, r)$, then $r - d(x, y) > 0$, and the triangle inequality gives

$$B_d(y, r - d(x, y)) \subseteq B_d(x, r).$$

Thus a subset is open exactly when it is a union of open balls.

Example 1.4. Euclidean space has the metric

$$d(x, y) = \|x - y\| = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}.$$

Unless otherwise specified, subsets of $\mathbb{R}^n$ carry the topology inherited from this metric.

Definition 1.5. A subset F of a topological space X is *closed* if $X \setminus F$ is open.

De Morgan's laws give the corresponding closed-set properties: $\emptyset$ and X are closed, arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed. A subset can be both open and closed, or neither.

1.3 Continuous maps

Definition 1.6. A map $f : X \rightarrow Y$ between topological spaces is *continuous* if $f^{-1}(V)$ is open in X for every open subset $V \subseteq Y$.

Here $f^{-1}(V) = \{x \in X : f(x) \in V\}$ denotes an inverse image; the map f need not be invertible. The definition concerns inverse images and does not require images of open sets to be open.

Proposition 1.7. *For metric spaces (X, d_X) and (Y, d_Y) , a map $f : X \rightarrow Y$ is continuous if and only if, for every $x \in X$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon.$$

Proof. Suppose first that f is continuous in the open-set sense. The inverse image of $B_{d_Y}(f(x), \varepsilon)$ is an open set containing x . It therefore contains some $B_{d_X}(x, \delta)$, which is precisely the required implication.

Conversely, assume the ε - δ condition. Let $V \subseteq Y$ be open and let $x \in f^{-1}(V)$. Choose $\varepsilon > 0$ with $B_{d_Y}(f(x), \varepsilon) \subseteq V$, and choose $\delta > 0$ from the assumed condition. Then $B_{d_X}(x, \delta) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is open. $\square$

Identity maps and constant maps are continuous. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then so is $g \circ f$, since

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)).$$

A map is continuous if and only if inverse images of closed sets are closed: use $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$.

1.4 Subspaces and products

We need topologies on subsets and products to formulate the examples and the definition of homotopy.

Definition 1.8. For a subset $A \subseteq X$, the *subspace topology* on A consists of all sets $A \cap U$ with U open in X .

The topology axioms follow by taking intersections with A in the axioms for X . Openness here is relative to A : for example, $[0, 1/2)$ is open in I , because it equals $I \cap (-1, 1/2)$. A subset of A is closed in A exactly when it has the form $A \cap F$ with F closed in X .

The inclusion $i : A \hookrightarrow X$ is continuous. A map $h : Z \rightarrow A$ is continuous if and only if $i \circ h : Z \rightarrow X$ is continuous, because

$$h^{-1}(A \cap U) = (i \circ h)^{-1}(U).$$

This lets us check continuity into a subspace in the ambient space, provided we first check that the values lie in the subspace.

Definition 1.9. The *product topology* on $X \times Y$ consists of arbitrary unions of sets $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open.

Finite intersections of open rectangles are open rectangles, so this is a topology. Both coordinate projections are continuous. A map $h : Z \rightarrow X \times Y$, written $h(z) = (h_1(z), h_2(z))$, is continuous exactly when h_1 and h_2 are continuous. One implication follows by composing with projections; for the other, use

$$h^{-1}(U \times V) = h_1^{-1}(U) \cap h_2^{-1}(V)$$

and take unions. In particular, this applies to $X \times I$, where I has its usual subspace topology in $\mathbb{R}$.

1.5 Homeomorphisms

Definition 1.10. A *homeomorphism* from X to Y is a bijection $f : X \rightarrow Y$ such that both f and f^{-1} are continuous. If such a map exists, the spaces are *homeomorphic*, written $X \cong Y$.

A homeomorphism identifies the topologies: a subset $U \subseteq X$ is open if and only if $f(U) \subseteq Y$ is open. Continuity of the inverse is part of the definition, not a consequence of continuity and bijectivity alone.

Example 1.11 (Euclidean space and the open ball). Let $B^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$. Define

$$f : B^n \longrightarrow \mathbb{R}^n, \quad f(x) = \frac{x}{1 - \|x\|}, \quad g : \mathbb{R}^n \longrightarrow B^n, \quad g(y) = \frac{y}{1 + \|y\|}.$$

The denominator of f is positive, and

$$\|g(y)\| = \frac{\|y\|}{1 + \|y\|} < 1,$$

so both maps are well defined and continuous. If $r = \|x\| < 1$, then

$$\|f(x)\| = \frac{r}{1 - r}, \quad g(f(x)) = \frac{x/(1 - r)}{1 + r/(1 - r)} = x.$$

Similarly, writing $\rho = \|y\|$ gives

$$f(g(y)) = \frac{y/(1 + \rho)}{1 - \rho/(1 + \rho)} = y.$$

Thus f and g are inverse homeomorphisms. Geometrically, f moves points along rays from the origin, replacing the radius r by $r/(1 - r)$. Radii approaching 1 are sent to arbitrarily large radii.

1.6 Connectedness and invariants

How can we show that two spaces are *not* homeomorphic? Trying and failing to construct a homeomorphism proves nothing. Instead, we look for a property preserved by every homeomorphism which differs between the spaces.

One such idea is *connectedness*: informally, whether a space consists of one piece or several separate pieces. A homeomorphism cannot join separate pieces or split a piece into two. For now we use this geometric intuition; precise definitions and the relevant proofs will come in Section 2.7 of Lecture 2.

Example 1.12 (The circle is not an interval). The circle S^1 and the interval I are each one piece, so this alone does not distinguish them. Instead, remove a point.

Removing any point p from the circle leaves an arc, still in one piece. Removing an interior point $a \in (0, 1)$ from the interval leaves two separate pieces:

$$I \setminus \{a\} = [0, a) \sqcup (a, 1].$$

If a homeomorphism from the circle to the interval existed, removing an interior point of the interval and its corresponding point on the circle would still leave homeomorphic spaces. But one would have two pieces and the other only one. The same argument applies to any real interval containing more than one point, by removing a point between two of its points.

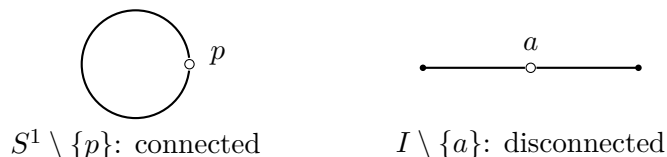


Figure 1: Connectedness after deleting a point distinguishes the circle from an interval. An open circle marks the deleted point.

This illustrates the method of invariants. Sometimes a basic invariant, such as the number of connected components, suffices. Sometimes we must apply it to related spaces, such as spaces obtained by removing points. Later, algebraic invariants will give more powerful ways to distinguish spaces.

1.7 Homotopy: a movie of maps

Fix two spaces X and Y . We want to express the idea that one continuous map $X \rightarrow Y$ can be continuously deformed into another.

Definition 1.13. Two continuous maps $f, g : X \rightarrow Y$ are *homotopic* if there exists a continuous map

$$h : X \times I \longrightarrow Y$$

such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for every $x \in X$. We call h a *homotopy from f to g* , and write $f \simeq g$.

Think of $s \in I$ as *time*. At time s the map is

$$h_s : X \rightarrow Y, \quad h_s(x) = h(x, s).$$

Thus a homotopy is a *movie of maps*: the first frame is f , the last frame is g , and the intervening frames describe the deformation. For a fixed $x \in X$, the curve $s \mapsto h(x, s)$ traces the motion of that point's image.

The requirement is joint continuity in (x, s) . Each frame is continuous, since $x \mapsto (x, s)$ is continuous, and each point traces a continuous path. These separate observations do not replace joint continuity of h .

Proposition 1.14. *Homotopy is an equivalence relation on the set of continuous maps from X to Y .*

Proof. Reflexivity. The stationary movie $h(x, s) = f(x)$ is a homotopy from f to itself.

Symmetry. If h is a homotopy from f to g , the reversed movie

$$\bar{h}(x, s) = h(x, 1 - s)$$

is a homotopy from g to f . It is continuous as a composition of continuous maps.

Transitivity. Suppose h_1 is a homotopy from f_0 to f_1 and h_2 is a homotopy from f_1 to f_2 . Run h_1 during the first half of the time interval and h_2 during the second half:

$$h(x, s) = \begin{cases} h_1(x, 2s), & 0 \leq s \leq \frac{1}{2}, \\ h_2(x, 2s - 1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

At the joining time the expressions agree: $h_1(x, 1) = f_1(x) = h_2(x, 0)$. Each formula is continuous on its own half, and the two halves are closed subsets covering $X \times I$. The pasting lemma below proves continuity of h . Its initial and final maps are f_0 and f_2 . $\square$

Lemma 1.15 (Pasting lemma for a finite closed cover). *Let $Z = A_1 \cup \dots \cup A_k$, where each A_i is closed in Z . If continuous maps $u_i : A_i \rightarrow Y$ agree on every overlap, the map $u : Z \rightarrow Y$ obtained by using u_i on A_i is continuous.*

Proof. Agreement on overlaps makes u well defined. For a closed subset $F \subseteq Y$, each $u_i^{-1}(F)$ is closed in A_i , and hence in Z because A_i is closed in Z . Thus

$$u^{-1}(F) = \bigcup_{i=1}^k u_i^{-1}(F)$$

is a finite union of closed subsets of Z , so is closed. $\square$

In the transitivity argument the closed subsets are $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$. They are inverse images of closed intervals under the projection to I . No separation assumption on X is needed.

1.8 Homotopy equivalence

Definition 1.16. Spaces X and Y are *homotopy equivalent* if there are continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ with

$$g \circ f \simeq \text{id}_X, \quad f \circ g \simeq \text{id}_Y.$$

The map f is a *homotopy equivalence*, and g a *homotopy inverse*. We write $X \simeq Y$ and say that X and Y have the same *homotopy type*.

For a homeomorphism the two compositions equal the identity maps. For a homotopy equivalence they need only be homotopic to the identity maps. Every homeomorphism is a homotopy equivalence, but a homotopy equivalence need not be a bijection.

The notation $\simeq$ has two related uses: between maps it means that the maps are homotopic; between spaces it means that they are homotopy equivalent.

Remark 1.17 (Compatibility with composition). If $f_0 \simeq f_1 : X \rightarrow Y$ and $a : W \rightarrow X$, $b : Y \rightarrow Z$ are continuous, then $b \circ f_0 \circ a \simeq b \circ f_1 \circ a$. A homotopy h between f_0 and f_1 gives the homotopy $(w, s) \mapsto b(h(a(w), s))$ between the composites.

Consequently, homotopy equivalence of spaces is itself an equivalence relation. Reflexivity and symmetry follow directly from the definition. For transitivity, suppose $f : X \rightarrow Y$ and $u : Y \rightarrow Z$ have homotopy inverses g and v . Then $g \circ v$ is a homotopy inverse to $u \circ f$:

$$(gv)(uf) = g(vu)f \simeq gf \simeq \text{id}_X, \quad (uf)(gv) = u(fg)v \simeq uv \simeq \text{id}_Z.$$

Composition symbols have been omitted in these chains.

1.9 Contractible spaces

Definition 1.18. A space X is *contractible* if it is homotopy equivalent to a one-point space $\{*\}$.

Proposition 1.19. A space X is contractible if and only if there exist $x_0 \in X$ and a homotopy $h : X \times I \rightarrow X$ with

$$h(x, 0) = x, \quad h(x, 1) = x_0 \quad \text{for every } x \in X.$$

Such a homotopy is called a *contraction* of X .

Proof. There is a unique map $p : X \rightarrow \{*\}$. A map $j : \{*\} \rightarrow X$ amounts to a choice of $x_0 = j(*)$. Both maps are continuous, and $p \circ j = \text{id}_{\{*\}}$, whereas $j \circ p$ is the constant map c_{x_0} . Thus p and j are homotopy inverses exactly when $\text{id}_X \simeq c_{x_0}$. $\square$

Contractibility requires X to be nonempty. A contraction moves every point to x_0 through points of X . The definition does not require x_0 to remain fixed at intermediate times; in our examples below it does.

Example 1.20 (Convex subsets of Euclidean space). A subset $C \subseteq \mathbb{R}^n$ is *convex* if $(1-s)x + sy \in C$ for all $x, y \in C$ and $s \in I$. Every nonempty convex subset is contractible. Choose $x_0 \in C$ and set

$$h : C \times I \longrightarrow C, \quad h(x, s) = (1-s)x + sx_0.$$

Convexity ensures that the values lie in C . The formula is continuous as a map into $\mathbb{R}^n$, hence as a map into the subspace C . Its endpoints are $h(x, 0) = x$ and $h(x, 1) = x_0$.

This contracts C along straight line segments. In particular, $\mathbb{R}^n$, open and closed balls, and nonempty intervals are contractible. For $n \geq 1$, $\mathbb{R}^n$ is not homeomorphic to a point, since no bijection exists. Thus homotopy equivalence is strictly weaker than homeomorphism.

1.10 The punctured plane and the circle

Let $X = \mathbb{R}^2 \setminus \{0\}$ and let $i : S^1 \hookrightarrow X$ be inclusion. Radial projection gives a continuous map

$$r : X \longrightarrow S^1, \quad r(x) = \frac{x}{\|x\|}.$$

It fixes points of the circle, so $r \circ i = \text{id}_{S^1}$. The other composite sends each nonzero vector to the unit vector in the same direction. It is homotopic to the identity by radial movement:

$$h : X \times I \longrightarrow X, \quad h(x, s) = (1-s)x + s \frac{x}{\|x\|} = \left(1-s + \frac{s}{\|x\|}\right)x. \quad (1.1)$$

The scalar in parentheses is strictly positive for $x \neq 0$ and $s \in I$, so the homotopy never passes through the omitted origin. The formula is continuous and satisfies

$$h(x, 0) = x, \quad h(x, 1) = i(r(x)).$$

Thus $i \circ r \simeq \text{id}_X$, and i and r are homotopy inverses. We have proved

$$\mathbb{R}^2 \setminus \{0\} \simeq S^1.$$

At time s , a point of radius ρ has radius $(1 - s)\rho + s$ and keeps its direction. Points outside the circle move inward, points inside move outward, and points on the circle remain fixed throughout.

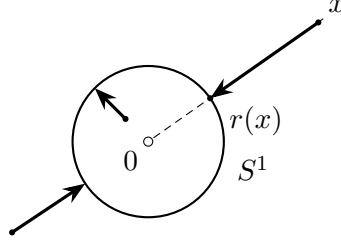


Figure 2: Radial deformation onto the unit circle. Every point travels along its ray, and no trajectory reaches the origin.

1.11 Retractions and deformation retractions

The preceding example has more structure than an arbitrary homotopy equivalence. One space is a subspace of the other, the map back fixes that subspace, and the deformation leaves it fixed at every time.

Definition 1.21. Let $A \subseteq X$ be a subspace, with inclusion $i : A \hookrightarrow X$. A *retraction* onto A is a continuous map $r : X \rightarrow A$ with $r(a) = a$ for all $a \in A$, or equivalently $r \circ i = \text{id}_A$. If such a map exists, A is called a *retract* of X .

Thus *retraction* names the map and *retract* names the subspace. A retraction alone does not provide a homotopy from id_X to $i \circ r$.

Definition 1.22. A *deformation retraction* of X onto a subspace A consists of a retraction $r : X \rightarrow A$ and a continuous map $h : X \times I \rightarrow X$ such that

$$h(x, 0) = x, \quad h(x, 1) = i(r(x)), \quad h(a, s) = a \quad (a \in A, s \in I).$$

When these data exist, A is a *deformation retract* of X .

Remark 1.23 (Terminology). We require A to remain fixed throughout the deformation. This is also called a *strong deformation retraction*. Some authors use “deformation retraction” without this requirement and reserve “strong” for the definition above. The convention here is always the one in Definition 1.22.

Proposition 1.24. If A is a deformation retract of X , then inclusion $i : A \rightarrow X$ is a homotopy equivalence, with homotopy inverse $r : X \rightarrow A$.

Proof. The retraction identity gives $r \circ i = \text{id}_A$. The deformation supplies $i \circ r \simeq \text{id}_X$. These are exactly the two conditions for a homotopy equivalence. $\square$

Both main examples have this form. In Example 1.20, $h(x, s) = (1 - s)x + sx_0$ fixes x_0 at every time, so $\{x_0\}$ is a deformation retract of the nonempty convex set C . In (1.1), if $a \in S^1$ then $\|a\| = 1$ and $h(a, s) = a$. Hence S^1 is a deformation retract of the punctured plane.

Deformation retractions provide a concrete way to prove homotopy equivalences: exhibit a subspace and a continuous motion of the whole space onto it, keeping that subspace fixed.

2 The homotopy category and the first invariants

Lecture 2, 7 October 2026

2.1 Categories: objects and morphisms

Our next goal is to interpret homotopy equivalence categorically. In a suitable category, saying that two spaces are homotopy equivalent will mean exactly that they are isomorphic. We first introduce the language needed to make this statement precise.

Definition 2.1. A *category* $\mathcal{C}$ consists of:

1. a collection of *objects*;
2. for each pair of objects X, Y , a set $\text{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms* from X to Y ;
3. for each triple X, Y, Z , a composition operation

$$\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(X, Z), \quad (g, f) \longmapsto g \circ f;$$

4. for every object X , an identity morphism $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$.

These data satisfy *associativity*,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

for every composable triple, and the *identity laws*,

$$\text{id}_Y \circ f = f = f \circ \text{id}_X \quad (f : X \rightarrow Y).$$

We write $f : X \rightarrow Y$ for a morphism with source X and target Y , and sometimes write gf for $g \circ f$: first apply f , then g . Objects may initially be thought of as “things,” with morphisms specifying the permitted ways to relate them. A morphism need not be a function between underlying sets, even though it is one in many familiar examples.

Remark 2.2 (Sets and classes). The objects of a category need not form a set. There is no set of all sets in ordinary set theory: unrestricted formation of such collections leads to Russell’s paradox. Indeed, a set R of all sets which do not contain themselves would satisfy $R \in R$ if and only if $R \notin R$. We therefore use the word *class* or *collection* when discussing all sets, all topological spaces, or all groups.

The definition above adopts the convention that each individual $\text{Hom}_{\mathcal{C}}(X, Y)$ is a set; this is called *local smallness*. All categories used here have this property. A category is *small* if its objects also form a set.

Example 2.3. The following are categories, with ordinary composition of maps and ordinary identity maps. In the vector-space example a field k is fixed.

Category	Objects	Morphisms
Top	Topological spaces	Continuous maps
Groups	Groups	Group homomorphisms
Vec_k	Vector spaces over k	k -linear maps
Set	Sets	Functions

For each row, compositions stay within the stated class of maps, identities belong to that class, and the category axioms follow from the corresponding laws for functions.

2.2 The homotopy category

In the first lecture we proved that homotopy is an equivalence relation on the continuous maps from X to Y . Write $[f]$ for the homotopy class of a continuous map $f : X \rightarrow Y$.

Definition 2.4. The *homotopy category* $\mathbf{HTop}$ has topological spaces as objects and

$$\mathrm{Hom}_{\mathbf{HTop}}(X, Y) = \{\text{continuous maps } X \rightarrow Y\} / \simeq$$

as its morphisms. Composition and identities are defined by

$$[g] \circ [f] = [g \circ f], \quad \mathrm{id}_X^{\mathbf{HTop}} = [\mathrm{id}_X].$$

We must verify that composition does not depend on the chosen representatives. We do this in two separate steps.

1. *Replacing f .* If $f \simeq f' : X \rightarrow Y$ and $g : Y \rightarrow Z$ is fixed, then $g \circ f \simeq g \circ f'$. Indeed, if $h : X \times I \rightarrow Y$ is a homotopy from f to f' , then

$$(x, s) \mapsto g(h(x, s))$$

is a homotopy from $g \circ f$ to $g \circ f'$. It is continuous as the composite $g \circ h$.

2. *Replacing g .* If $g \simeq g' : Y \rightarrow Z$ and $f : X \rightarrow Y$ is fixed, then $g \circ f \simeq g' \circ f$. Indeed, if $h : Y \times I \rightarrow Z$ is a homotopy from g to g' , then

$$(x, s) \mapsto h(f(x), s)$$

is a homotopy from $g \circ f$ to $g' \circ f$. It is continuous as the composite $h \circ (f \times \mathrm{id}_I)$.

Consequently, if both representatives are replaced, transitivity gives

$$g \circ f \simeq g \circ f' \simeq g' \circ f'.$$

Thus $[g \circ f] = [g' \circ f']$, so composition is well defined.

The remaining axioms descend directly from composition of continuous maps:

$$[h] \circ ([g] \circ [f]) = [h \circ (g \circ f)] = [(h \circ g) \circ f] = ([h] \circ [g]) \circ [f],$$

and

$$[\mathrm{id}_Y] \circ [f] = [f] = [f] \circ [\mathrm{id}_X].$$

This completes the verification that $\mathbf{HTop}$ is a category. Its objects are the same as those of $\mathbf{Top}$; it is the morphisms which have changed. Two homotopic maps become the same morphism.

2.3 Isomorphisms in a category

Definition 2.5. A morphism $f : X \rightarrow Y$ in a category is an *isomorphism* if there exists a morphism $g : Y \rightarrow X$ such that

$$gf = \mathrm{id}_X, \quad fg = \mathrm{id}_Y.$$

Then g is an *inverse* of f , denoted f^{-1} , and the objects X, Y are *isomorphic*.

For example, isomorphisms in $\mathbf{Top}$ are homeomorphisms, isomorphisms in $\mathbf{Groups}$ are group isomorphisms, and isomorphisms in $\mathbf{Vec}_k$ are invertible linear maps.

Lemma 2.6 (Left and right inverses). *If $f : X \rightarrow Y$ has a left inverse $\ell : Y \rightarrow X$ and a right inverse $r : Y \rightarrow X$, then $\ell = r$ and f is an isomorphism. In particular, the inverse of an isomorphism is unique.*

Proof. The assumptions are $\ell f = \text{id}_X$ and $fr = \text{id}_Y$. Associativity gives

$$\ell = \ell \text{id}_Y = \ell(fr) = (\ell f)r = \text{id}_X r = r.$$

Thus the common morphism is a two-sided inverse. Applying the same argument to any two inverses proves uniqueness. $\square$

Lemma 2.7 (Two out of three). *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be composable morphisms. If any two of f , g , and gf are isomorphisms, so is the third.*

Proof. If f and g are isomorphisms, then $(gf)^{-1} = f^{-1}g^{-1}$: composing in either order gives the appropriate identity.

If f and gf are isomorphisms, write $g = (gf)f^{-1}$. This is a composite of isomorphisms, so is an isomorphism; its inverse is $f(gf)^{-1}$.

If g and gf are isomorphisms, write $f = g^{-1}(gf)$. Its inverse is $(gf)^{-1}g$. $\square$

2.4 Homotopy equivalence as isomorphism

We can now return to the motivating question.

Proposition 2.8. *A continuous map $f : X \rightarrow Y$ is a homotopy equivalence if and only if $[f] : X \rightarrow Y$ is an isomorphism in **HTop**.*

Proof. An inverse to $[f]$ is represented by a continuous map $g : Y \rightarrow X$. The inverse identities say

$$[g][f] = [\text{id}_X], \quad [f][g] = [\text{id}_Y].$$

By definition of equality of homotopy classes, these say precisely

$$gf \simeq \text{id}_X, \quad fg \simeq \text{id}_Y.$$

Thus g represents an inverse exactly when it is a homotopy inverse of f . $\square$

This is the promised categorical interpretation: homotopy equivalence is simply isomorphism in **HTop**. The isomorphism lemmas therefore apply to homotopy equivalences. In particular, homotopy equivalences satisfy the two-out-of-three property.

Isomorphic objects are indistinguishable from the point of view of their category. The qualification “in their category” matters: a nonempty convex subset of $\mathbb{R}^n$ and a point are isomorphic in **HTop**, even though they usually are not homeomorphic. Likewise, the punctured plane and the circle are isomorphic in **HTop** by the radial deformation constructed in Lecture 1.

Here is a precise way to use this principle. If $u : X \rightarrow Y$ is an isomorphism in any category and A is an object, then composition with u gives a bijection

$$\text{Hom}_C(A, X) \longrightarrow \text{Hom}_C(A, Y), \quad a \longmapsto u \circ a. \quad (2.1)$$

Its inverse is composition with u^{-1} . We will apply this first with A a point, and then with A a circle, in **HTop**.

2.5 Paths and path components

Definition 2.9. A *path* from x to y in a space X is a continuous map $\gamma : I \rightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$. A space is *path-connected* if any two of its points can be joined by a path.

In particular, the empty space is path-connected: the condition is vacuous. Paths can be reversed and joined together. If γ runs from x to y , $\gamma^{-1}(t) = \gamma(1 - t)$ runs from y to x . If δ runs from y to z , their concatenation is

$$(\gamma\delta)(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq \frac{1}{2}, \\ \delta(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The formulas agree at $t = 1/2$, so the pasting lemma (Lemma 1.15) proves continuity. Here $\gamma\delta$ means first traverse γ , then δ .

Lemma 2.10. *The relation “can be joined by a path” is an equivalence relation on X .*

Proof. Constant paths give reflexivity, reversal gives symmetry, and concatenation gives transitivity. $\square$

Definition 2.11. The *path component* of $x \in X$ is

$$C_x = \{y \in X : \text{there is a path from } x \text{ to } y\}.$$

The set of all path components is denoted

$$\pi_0(X) = \{C_x : x \in X\}.$$

As equivalence classes, any two sets C_x, C_y are either equal or disjoint, and together they partition X . Each C_x is itself path-connected: paths from x to points of C_x stay inside C_x , since every intermediate point can be reached by restricting and reparametrizing the path. Reversal and concatenation then join any two points of C_x within C_x .

The notation $\pi_0(X)$ means a *set*, not just the number of its elements. A space is path-connected exactly when this set has at most one element: zero if the space is empty, and one otherwise.

2.6 The invariant π_0

Let $*$ denote a one-point space. A continuous map $* \rightarrow X$ is exactly a choice of a point $x \in X$; write c_x for this map. A homotopy $* \times I \rightarrow X$ between c_x and c_y is exactly a path from x to y . We therefore have a canonical bijection

$$\text{Hom}_{\mathbf{HTop}}(*, X) \longrightarrow \pi_0(X), \quad [c_x] \longmapsto C_x. \quad (2.2)$$

It is well defined and injective because $c_x \simeq c_y$ exactly when $C_x = C_y$, and it is surjective because every path component contains a point.

A continuous map $f : X \rightarrow Y$ sends paths to paths by composition, hence induces a map

$$\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y), \quad C_x \longmapsto C_{f(x)}.$$

This does not assert that $f(C_x)$ is the whole component of $f(x)$; it says that $f(C_x)$ is contained in that component. Under (2.2), this construction is composition with $[f]$. It follows that

$$\pi_0(\text{id}_X) = \text{id}_{\pi_0(X)}, \quad \pi_0(gf) = \pi_0(g) \circ \pi_0(f).$$

Moreover, homotopic maps induce the same map on components. Directly, if h is a homotopy from f to g , then $s \mapsto h(x, s)$ is a path from $f(x)$ to $g(x)$, so $C_{f(x)} = C_{g(x)}$.

Proposition 2.12. *A homotopy equivalence $f : X \rightarrow Y$ induces a bijection $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$.*

Proof. If g is a homotopy inverse, the preceding identities give

$$\pi_0(g)\pi_0(f) = \pi_0(gf) = \pi_0(\text{id}_X), \quad \pi_0(f)\pi_0(g) = \pi_0(fg) = \pi_0(\text{id}_Y).$$

Thus $\pi_0(g)$ is an inverse. Equivalently, apply (2.1) in **HTop** with $A = *$. $\square$

Consequently, homotopy equivalent spaces have the same number of path components, whether finite or infinite. Path-connectedness is a homotopy invariant. In particular, a contractible space is path-connected. The action on maps is part of the invariant: π_0 does more than assign an unrelated set to each space. Assignments preserving identities and composition in this way are called *functors*; thus π_0 can be viewed as a functor from **HTop** to **Set**.

2.7 Connectedness and local path-connectedness

We now make precise the connectedness discussion from Lecture 1. Connectedness concerns separation by open sets; path-connectedness instead asks for actual continuous paths.

Definition 2.13. A space X is *connected* if it cannot be expressed as a union $X = U \cup V$ of two disjoint, nonempty open subsets. Such a decomposition is called a *separation*. A subset of X is connected if it is connected in its subspace topology.

Definition 2.14 (Connected component). A *connected component* of a space X is a nonempty subset $C \subseteq X$ which is *open in X* , *closed in X* , and *connected* in its subspace topology.

Terminology. This is the convention of this course. Standard terminology calls any maximal nonempty connected subset a connected component; such subsets need not be open. Our components are exactly the open standard components. In an arbitrary space, they need not cover all of X .

Proposition 2.15. *Any two connected components of a space either are disjoint or coincide.*

Proof. Let C_1, C_2 be connected components with $C_1 \cap C_2 \neq \emptyset$. Since C_2 is open and closed in X , the sets $C_1 \cap C_2$ and $C_1 \setminus C_2$ are disjoint open subsets of C_1 whose union is C_1 . Connectedness of C_1 forces the second set to be empty, so $C_1 \subseteq C_2$. Interchanging C_1 and C_2 gives the reverse inclusion. Hence $C_1 = C_2$. $\square$

Lemma 2.16. *The continuous image of a connected space is connected.*

Proof. Let $f : X \rightarrow Y$ be continuous, with X connected, and regard f as a map onto the subspace $f(X)$. A separation of $f(X)$ would pull back to two disjoint, nonempty open subsets covering X , a contradiction. $\square$

A homeomorphism therefore preserves connectedness. It also preserves openness, closedness, and nonemptiness, so it sends connected components in our convention bijectively to connected components. Hence their number is a topological invariant.

Intervals in $\mathbb{R}$ are connected. Indeed, if an interval J had a separation $U \cup V$, the function equal to 0 on U and 1 on V would be continuous, since it is constant on an open neighbourhood of each point. Between a point of U and a point of V , this contradicts the intermediate value theorem.

For example, the continuous surjection

$$(0, 2\pi) \longrightarrow S^1 \setminus \{(1, 0)\}, \quad t \longmapsto (\cos t, \sin t)$$

shows that the punctured circle is connected. Rotating the circle gives the same conclusion for any deleted point. This justifies the geometric observation in Example 1.12.

Proposition 2.17. *Every path-connected space is connected.*

Proof. Suppose a path-connected space X had a separation $X = U \cup V$. Choose $x \in U$, $y \in V$, and a path γ from x to y . Since I is connected, Lemma 2.16 implies that $\gamma(I)$ is connected. But its intersections with U and V are disjoint nonempty relatively open subsets covering $\gamma(I)$, a contradiction. $\square$

The converse is false: a connected space need not be path-connected.

Example 2.18 (The topologist's sine curve). Let

$$G = \{(x, \sin(1/x)) : 0 < x \leq 1\}, \quad V = \{0\} \times [-1, 1], \quad T = G \cup V.$$

The graph oscillates increasingly rapidly as x approaches zero, accumulating on the whole vertical interval V . The space T is connected but not path-connected.

Connectedness. The graph G is the continuous image of the connected interval $(0, 1]$, so G is connected. Its closure in $\mathbb{R}^2$ is exactly T . Indeed, for each $y \in [-1, 1]$, choose θ with $\sin \theta = y$. For all sufficiently large n ,

$$x_n = \frac{1}{\theta + 2\pi n} \in (0, 1], \quad (x_n, \sin(1/x_n)) = (x_n, y) \longrightarrow (0, y).$$

Thus every point of V is a limit point of G . There are no other new limit points: for a positive limiting first coordinate, continuity of $\sin(1/x)$ keeps the limit on G , and all second coordinates lie in the closed interval $[-1, 1]$.

The closure of a connected set is connected. To see this, a separation of the closure would have two nonempty relatively open sides, both of which meet the dense original set; their intersections would separate that set. Hence $T = \overline{G}$ is connected.

Failure of path-connectedness. Suppose a path $\gamma : I \rightarrow T$, written $\gamma(t) = (x(t), y(t))$, starts in V and ends in G . Let

$$a = \max\{t \in I : x(t) = 0\}.$$

This maximum exists because the set is nonempty and closed in I . We have $a < 1$ and $x(t) > 0$ for every $t > a$.

By continuity of y at a , choose b with $a < b \leq 1$ so close to a that

$$|y(t) - y(a)| < \frac{1}{2} \quad (a \leq t \leq b).$$

But $x(a) = 0$ and $x(b) > 0$. The intermediate value theorem implies that x takes every value between 0 and $x(b)$ on $[a, b]$. Among these positive values are numbers u, v as small as desired with $\sin(1/u) = 1$ and $\sin(1/v) = -1$. At times when $x(t)$ equals u or v , the path lies in G , so $y(t)$ equals 1 or -1 . Both values cannot be within $1/2$ of $y(a)$, a contradiction.

Each of G and V is path-connected: G is homeomorphic to $(0, 1]$ by projection onto the first coordinate, and V is an interval. They are therefore precisely the two path components of T . On the other hand, T itself is its single connected component. This illustrates the difference between the two notions.

Definition 2.19. A space X is *locally path-connected* if for every $x \in X$ and every open neighbourhood U of x , there is a path-connected open subset V of X with $x \in V \subseteq U$.

Thus each point must have arbitrarily small path-connected open neighbourhoods. Having just one such neighbourhood is insufficient: a path-connected space always provides a large one, namely itself. Euclidean spaces are locally path-connected because open balls form a neighbourhood basis and are convex.

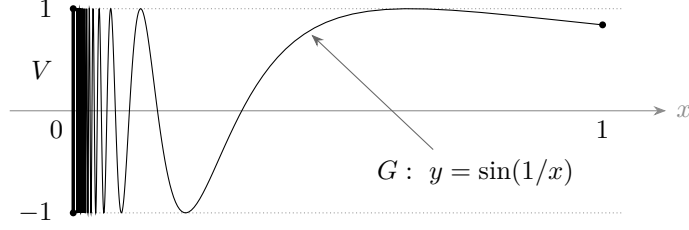


Figure 3: The topologist's sine curve includes the limiting vertical interval. The oscillations continue arbitrarily close to $x = 0$; only a finite portion of the graph is drawn.

Example 2.20 (The comb). Consider the subspace

$$K = (I \times \{0\}) \cup (\{0\} \times I) \cup \bigcup_{n \geq 1} (\{1/n\} \times I) \subset \mathbb{R}^2.$$

This is a horizontal base with vertical teeth at 0 and at each $1/n$. It is path-connected: travel vertically down to the base, along the base, and vertically up to the desired point. Each segment stays inside K , and concatenating the segments gives a continuous path.

It is not locally path-connected at any $p = (0, a)$ with $0 < a \leq 1$. Take the open neighbourhood

$$U = K \cap \{(u, v) \in \mathbb{R}^2 : v > a/2\}.$$

A path lying in U must have constant first coordinate. Indeed, that coordinate takes values only in $\{0\} \cup \{1/n : n \geq 1\}$, which contains no interval of positive length. The intermediate value theorem therefore forces it to be constant. So no path in U joins p to a point of a tooth at $1/n$. But every open neighbourhood V of p in K contains $(1/n, a)$ for sufficiently large n , since these points converge to p . Consequently no such $V \subseteq U$ can be path-connected.

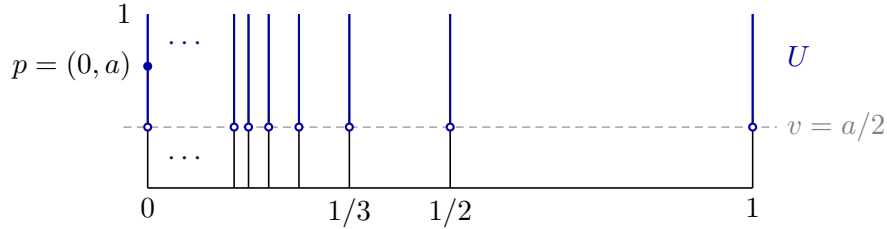


Figure 4: The comb K : the dots indicate the remaining teeth at $1/n$, accumulating at the included vertical segment over 0. Blue shows U ; the hollow markers belong to K but not to U . The dashed horizontal line is only a guide, not part of the comb.

Proposition 2.21. *In a locally path-connected space, path components are open and closed, and they coincide with connected components.*

Proof. Let C be a path component. For each $x \in C$, local path-connectedness gives a path-connected open neighbourhood V_x of x . Every point of V_x can be joined to x , so $V_x \subseteq C$. Thus C is open. All path components are open, and they partition X , so $X \setminus C$ is a union of open components. Hence C is also closed.

The set C is nonempty and connected, so it is a connected component by our definition.

Conversely, let K be a connected component and choose $x \in K$. Let C be the path component of x . Since K is open and closed, connectedness of C forces $C \subseteq K$. Since C is open and closed, connectedness of K forces $K \subseteq C$. Hence $K = C$. The path components are therefore exactly the connected components. $\square$

In particular, a connected, locally path-connected space is path-connected. The local hypothesis is essential to this argument. For locally path-connected spaces, the connected components in our convention also agree with the standard connected components: the path components partition the space into connected open-and-closed subsets, each of which is maximal connected.

2.8 Free loops and simple connectedness

The next object we would like to understand is π_1 . Following the construction of π_0 , it is natural to look at

$$\mathrm{Hom}_{\mathbf{HTop}}(S^1, X).$$

Its elements are *free homotopy classes of loops*: a loop here is a continuous map $S^1 \rightarrow X$, and homotopies impose no condition on a chosen point of the circle. This is a useful invariant, but this construction alone gives a set, without the specified basepoint and multiplication needed for the fundamental group.

Before imposing a basepoint, we can formulate simple connectedness.

Definition 2.22. A space X is *simply connected* if it is path-connected and every continuous map $f : S^1 \rightarrow X$ is homotopic to a constant map. A map homotopic to a constant map is called *null-homotopic*.

The homotopies in this definition are free homotopies. The path-connectedness requirement is part of the definition. Connectedness alone does not substitute for it in arbitrary spaces. The empty space is simply connected: it is path-connected and has no loops, so the condition on loops is vacuous.

Proposition 2.23. *A space X is simply connected if and only if $\mathrm{Hom}_{\mathbf{HTop}}(S^1, X)$ has at most one element.*

Proof. If $X = \emptyset$, both conditions hold. Assume now that $X \neq \emptyset$.

Suppose X is simply connected. Every loop is homotopic to a constant. Any two constant maps $c_x, c_y : S^1 \rightarrow X$ are homotopic: if γ is a path from x to y , use $h(z, s) = \gamma(s)$. Thus all loops represent the same class, and at least one loop exists because X is nonempty.

Conversely, suppose the Hom-set has at most one element. Since X is nonempty, a constant map exists, so it has exactly one element. For any $x, y \in X$, the constant maps c_x and c_y are homotopic. Evaluating such a homotopy at a fixed point of S^1 gives a path from x to y , so X is path-connected. Every loop is homotopic to a constant because there is just one homotopy class. $\square$

Remark 2.24. For every nonempty X a constant map $S^1 \rightarrow X$ exists, so this Hom-set cannot have zero elements. If $X = \emptyset$, the Hom-set is empty. Thus a simply connected space has zero free homotopy classes of loops if it is empty, and exactly one otherwise.

Corollary 2.25. *Simple connectedness is a homotopy invariant.*

Proof. A homotopy equivalence $X \simeq Y$ induces a bijection

$$\mathrm{Hom}_{\mathbf{HTop}}(S^1, X) \longrightarrow \mathrm{Hom}_{\mathbf{HTop}}(S^1, Y)$$

by composition, as in (2.1). One of these sets has at most one element if and only if the other does. $\square$

Example 2.26. A point is simply connected. Therefore every contractible space is simply connected. In particular, every nonempty convex subset $C \subseteq \mathbb{R}^n$ is simply connected. Directly, line segments join its points, and a loop $f : S^1 \rightarrow C$ contracts to a chosen $x_0 \in C$ by

$$h(z, s) = (1 - s)f(z) + sx_0.$$

Remark 2.27 (Results announced for later proof). The circle S^1 is not simply connected, and consequently neither is $\mathbb{R}^2 \setminus \{0\}$, since it is homotopy equivalent to S^1 . The higher-dimensional spheres S^n , $n \geq 2$, are simply connected. These statements are announced here; their proofs require further tools and are not supplied in this lecture.

2.9 Based loops and $\pi_1(X, x_0)$ as a set

To prepare for the fundamental group, we choose a point $x_0 \in X$. We use the complex model

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

with distinguished point $1 \in S^1$.

Definition 2.28. A *based loop* in (X, x_0) is a continuous map $f : S^1 \rightarrow X$ with $f(1) = x_0$. Two based loops f, g are *based homotopic* if there is a continuous map $h : S^1 \times I \rightarrow X$ with

$$h(z, 0) = f(z), \quad h(z, 1) = g(z), \quad h(1, s) = x_0 \quad \text{for all } s \in I.$$

In the movie interpretation, every frame must remain a based loop: the image of 1 stays fixed at x_0 throughout. This is stronger than merely requiring the first and last frames to be based. During a free homotopy, that distinguished point is allowed to move.

Based homotopy is an equivalence relation. The stationary, reversed, and concatenated homotopies used in Proposition 1.14 all preserve the condition $h(1, s) = x_0$ when the original homotopies do. We may therefore form the quotient set

$$\pi_1(X, x_0) = \{f : S^1 \rightarrow X \text{ continuous} : f(1) = x_0\} / \text{based homotopy}. \quad (2.3)$$

The constant based loop $c_{x_0}(z) = x_0$ supplies a distinguished element. A based loop f also gives a path

$$\gamma_f(t) = f(e^{2\pi it}), \quad 0 \leq t \leq 1,$$

which starts and ends at x_0 .

Forgetting the basepoint condition gives a well-defined map

$$\pi_1(X, x_0) \longrightarrow \mathrm{Hom}_{\mathbf{HTop}}(S^1, X), \quad [f]_{\text{based}} \longmapsto [f]_{\text{free}},$$

because every based homotopy is a free homotopy. The two quotients use different equivalence relations and should not be identified without further argument.

At this stage, $\pi_1(X, x_0)$ has been defined as a set of based homotopy classes. A multiplication and the proof of the group axioms remain to be constructed.

3 The fundamental group, coverings, and degree

Lecture 3, 8 October 2026

3.1 Based loops and their multiplication

We now give the set $\pi_1(X, x_0)$ from Lecture 2 a group structure. Recall that a based loop was defined as a continuous map

$$f : S^1 \longrightarrow X, \quad f(1) = x_0,$$

We identify the circle with the interval whose endpoints are glued, using the homeomorphism

$$I/(0 \sim 1) \longrightarrow S^1, \quad [t] \longmapsto e^{2\pi it}.$$

The identified endpoints $[0] = [1]$ correspond to the point $1 \in S^1$.

From now on, we write a based loop as a continuous map

$$\gamma : I \longrightarrow X, \quad \gamma(0) = \gamma(1) = x_0.$$

A *based homotopy* from γ to δ is a continuous map $h : I \times I \rightarrow X$ satisfying

$$h(t, 0) = \gamma(t), \quad h(t, 1) = \delta(t), \quad h(0, s) = h(1, s) = x_0.$$

We write $\gamma \simeq_* \delta$. The horizontal coordinate t runs along the loop; the vertical coordinate s is homotopy time. Both vertical sides of the parameter square must stay at x_0 .

For two based loops, define their *concatenation* by

$$(\gamma\delta)(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq \frac{1}{2}, \\ \delta(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases} \quad (3.1)$$

Thus $\gamma\delta$ travels along γ first and δ second. The formulas agree at $t = \frac{1}{2}$, since both give x_0 . The pasting lemma therefore proves continuity.

Proposition 3.1. *The rule $[\gamma][\delta] = [\gamma\delta]$ is a well-defined operation on $\pi_1(X, x_0)$.*

Proof. We check independence of the two representatives separately.

1. *Replacing γ .* Suppose $\gamma \simeq_* \gamma'$ and fix δ . If h is a based homotopy from γ to γ' , then

$$(t, s) \longmapsto \begin{cases} h(2t, s), & 0 \leq t \leq \frac{1}{2}, \\ \delta(2t - 1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a based homotopy from $\gamma\delta$ to $\gamma'\delta$.

2. *Replacing δ .* Suppose $\delta \simeq_* \delta'$ and fix γ . If h is a based homotopy from δ to δ' , then

$$(t, s) \longmapsto \begin{cases} \gamma(2t), & 0 \leq t \leq \frac{1}{2}, \\ h(2t - 1, s), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a based homotopy from $\gamma\delta$ to $\gamma\delta'$.

In each case the formulas agree along $t = \frac{1}{2}$, where both equal x_0 . Continuity follows by pasting on the two closed rectangles, and both vertical sides stay at x_0 . If both representatives change, transitivity gives

$$\gamma\delta \simeq_* \gamma'\delta \simeq_* \gamma'\delta'.$$

□

3.2 The group axioms as square homotopies

Let $e(t) = x_0$ be the constant loop and let $\gamma^{-1}(t) = \gamma(1-t)$ be the reversed loop. Concatenation changes the speed of travel and introduces pauses. The group laws assert that these changes disappear on based homotopy classes.

The identity. The loop $e\gamma$ waits at x_0 for half the time and then traverses γ . Shrink the waiting interval from length $\frac{1}{2}$ to zero. Put $a(s) = (1-s)/2$ and define

$$h(t, s) = \begin{cases} x_0, & 0 \leq t \leq a(s), \\ \gamma\left(\frac{t-a(s)}{1-a(s)}\right), & a(s) \leq t \leq 1. \end{cases} \quad (3.2)$$

The denominator is always at least $1/2$. The regions are closed and the formulas agree on their common boundary, so h is continuous. Its bottom edge is $e\gamma$, its top edge is γ , and its two vertical sides have value x_0 . Thus $e\gamma \simeq_* \gamma$.

For the right identity, put $b(s) = (1+s)/2$ and use

$$h(t, s) = \begin{cases} \gamma(t/b(s)), & 0 \leq t \leq b(s), \\ x_0, & b(s) \leq t \leq 1. \end{cases}$$

This removes the pause at the end and gives $\gamma e \simeq_* \gamma$.

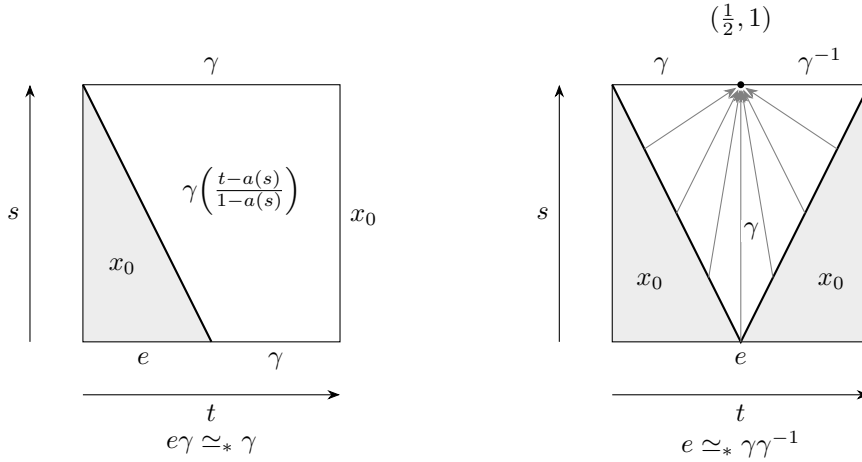


Figure 5: Parameter squares for the identity and inverse laws, redrawn from the blackboard constructions. Read each horizontal slice as a loop. In the inverse square, each arrow carries γ towards $(\frac{1}{2}, 1)$; the two outer triangles are constant at x_0 .

Inverses. We construct a based homotopy from e to $\gamma\gamma^{-1}$. In the parameter square, draw the triangle with vertices $(0, 1)$, $(\frac{1}{2}, 0)$, and $(1, 1)$. Put x_0 on the two outer right triangles. Fill the middle triangle by placing γ on every segment from its sloping boundary to $(\frac{1}{2}, 1)$, with parameter 0 at the boundary and 1 at $(\frac{1}{2}, 1)$.

A formula for this construction is

$$h(t, s) = \gamma(\max\{0, s - |2t - 1|\}). \quad (3.3)$$

Inside the middle triangle, $s - |2t - 1|$ runs linearly from 0 to 1 along each such segment; outside it the maximum is 0. The formula is continuous and satisfies

$$h(t, 0) = x_0, \quad h(t, 1) = \gamma(1 - |2t - 1|) = (\gamma\gamma^{-1})(t),$$

as well as $h(0, s) = h(1, s) = x_0$. Hence $e \simeq_* \gamma\gamma^{-1}$. Applying the same construction to γ^{-1} , with $(\gamma^{-1})^{-1} = \gamma$, gives $e \simeq_* \gamma^{-1}\gamma$. Also, reversing a based homotopy h by replacing t with $1 - t$ shows that $[\gamma^{-1}]$ depends only on $[\gamma]$.

Associativity. The loops $(\gamma_1\gamma_2)\gamma_3$ and $\gamma_1(\gamma_2\gamma_3)$ traverse the same three loops in the same order. Their time allocations differ:

	γ_1	γ_2	γ_3
$(\gamma_1\gamma_2)\gamma_3$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$\gamma_1(\gamma_2\gamma_3)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Move the two breakpoints linearly, setting $a(s) = (1 + s)/4$ and $b(s) = (2 + s)/4$. Define

$$h(t, s) = \begin{cases} \gamma_1(t/a(s)), & 0 \leq t \leq a(s), \\ \gamma_2((t - a(s))/(b(s) - a(s))), & a(s) \leq t \leq b(s), \\ \gamma_3((t - b(s))/(1 - b(s))), & b(s) \leq t \leq 1. \end{cases} \quad (3.4)$$

All denominators are positive. On each seam both formulas give x_0 , so the finite closed-cover pasting lemma proves continuity. The bottom and top edges give the two parenthesizations, and the vertical sides stay at x_0 . This is the required based homotopy.

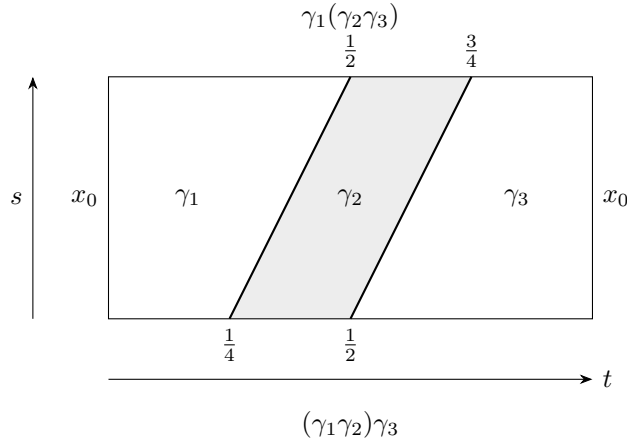


Figure 6: Associativity changes the time assigned to each loop. Within each region, traverse the indicated loop once as t crosses that region.

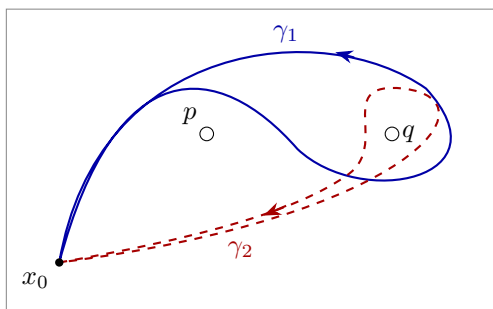
Theorem 3.2. *For every pointed space (X, x_0) , concatenation makes $\pi_1(X, x_0)$ into a group, called the fundamental group. Its identity is the class of the constant loop, and $[\gamma]^{-1} = [\gamma^{-1}]$.*

Proof. The operation is well defined by Proposition 3.1. The square homotopies above prove both identity laws, both inverse laws, and associativity. $\square$

3.3 Why free homotopy does not suffice

Consider the loops γ_1, γ_2 in the rectangle with punctures p, q shown in Figure 7. They are freely homotopic: if the basepoint may move, each can be deformed to the same loop around q .

Convince yourself from the picture that $\gamma_1\gamma_2^{-1}$ is not homotopic to a constant map. In contrast, $\gamma_2\gamma_2^{-1} \simeq_* e$. Thus replacing γ_1 by the freely homotopic loop γ_2 changes the homotopy class of the product. This is why we use based homotopies to define multiplication.



$\gamma_1 \simeq \gamma_2$ freely, but
 $\gamma_1 \gamma_2^{-1} \not\simeq e$.

Figure 7: Two freely homotopic loops based at x_0 in a twice-punctured rectangle. Follow the arrows for their orientations.

The picture-hanging problem. How can a picture be hung by a string over two nails so that it stays up with both nails in place, but falls if either one is removed? Use $\gamma_1 \gamma_2^{-1}$ as a guide: check that this loop becomes homotopic to a constant when either puncture is filled in.

3.4 Covering maps: local copies of the base

Our next goal is the computation

$$\pi_1(S^1, 1) \cong \mathbb{Z}.$$

We begin with the map

$$f : \mathbb{R} \longrightarrow S^1, \quad f(x) = (\cos(2\pi x), \sin(2\pi x)) = e^{2\pi i x}. \quad (3.5)$$

One complete positive turn corresponds to an increase of 1 in the real coordinate. Although f is not globally one-to-one, a small arc of the circle has disjoint copies in $\mathbb{R}$, one in each turn. This is the local picture of a covering map.

Definition 3.3. A continuous map $f : X \rightarrow Y$ is a *covering map* if every $y \in Y$ has an open neighborhood U such that

$$f^{-1}(U) = \coprod_{\alpha \in A} V_\alpha,$$

where the V_α are pairwise disjoint open subsets of X and each restriction $f|_{V_\alpha} : V_\alpha \rightarrow U$ is a homeomorphism. The index set A may be empty; surjectivity is not required. The space Y is the *base*, X is the *cover* or *covering space*, and U is an *evenly covered neighborhood*. The subsets V_α are its *sheets*.

To read the definition, first choose one small open set U downstairs. Then examine its *entire* inverse image upstairs. This inverse image must split into disjoint open pieces, each mapped bijectively and homeomorphically onto the whole of U . After identifying each piece with U by this homeomorphism, the restriction of f becomes the identity. The sheets are local: they need not extend to disjoint copies of all of Y inside X .

Example 3.4 (The real line covering the circle). Choose $a \in \mathbb{R}$ and $0 < \varepsilon < 1/2$. The image $U = f((a - \varepsilon, a + \varepsilon))$ is an open arc containing $f(a)$, and

$$f^{-1}(U) = \coprod_{n \in \mathbb{Z}} (a - \varepsilon + n, a + \varepsilon + n).$$

On each interval, f is a homeomorphism onto U : its inverse chooses the continuous angle along that arc, in the indicated interval. These intervals are disjoint because their length is less than 1. Thus (3.5) is a covering map. Notice that $f(x) = f(x')$ if and only if $x - x' \in \mathbb{Z}$.

Example 3.5 (Two disks over one disk). Let $D = \{u \in \mathbb{R}^2 : \|u\| < 1\}$ and give $\{1, 2\}$ the discrete topology. Then

$$f : D \times \{1, 2\} \longrightarrow D, \quad f(u, j) = u$$

is a covering. We may take $U = D$ itself: its inverse image has the two open sheets $D \times \{1\}$ and $D \times \{2\}$. This example consists of two separate copies of the whole base.

Example 3.6 (The double covering of the circle). The map $f : S^1 \rightarrow S^1$, $f(z) = z^2$, is a covering with two sheets over each small arc. To see this, write that arc as

$$U = \{e^{2\pi it} : a - \varepsilon < t < a + \varepsilon\}, \quad 0 < \varepsilon < 1/2.$$

Its inverse image is the disjoint union of the arcs

$$V_0 = \{e^{\pi it} : a - \varepsilon < t < a + \varepsilon\}, \quad V_1 = -V_0.$$

The two inverse branches are $e^{2\pi it} \mapsto e^{\pi it}$ and $e^{2\pi it} \mapsto -e^{\pi it}$, using the unique t in the chosen interval. Each branch is continuous. In contrast to the two-disk example, this covering space is connected: the local sheets belong to a single circle.

Example 3.7 (A projection which is not a covering). The projection $p : \mathbb{R}^2 \rightarrow \mathbb{R}$, $p(u, v) = u$, is not a covering map. For a nonempty open interval $U \subset \mathbb{R}$, the inverse image $p^{-1}(U) = U \times \mathbb{R}$ is connected but is not homeomorphic to U .

3.5 Lifting paths and homotopies

A *lift* of a map $g : Z \rightarrow Y$ through $f : X \rightarrow Y$ is a continuous map $\tilde{g} : Z \rightarrow X$ such that $f \circ \tilde{g} = g$. It makes the same motion upstairs that g describes downstairs. For the exponential covering, lifting means choosing a continuous real-valued angle, measured in turns.

Theorem 3.8 (Homotopy lifting). *Let $f : X \rightarrow Y$ be a covering map, let $h : Z \times I \rightarrow Y$ be a continuous homotopy, and let $\alpha : Z \rightarrow X$ be a continuous initial lift, so that $f(\alpha(z)) = h(z, 0)$. There is a unique continuous map $\tilde{h} : Z \times I \rightarrow X$ such that*

$$f \circ \tilde{h} = h, \quad \tilde{h}(z, 0) = \alpha(z).$$

The proof will be given in a later lecture.

The data and the conclusion have different roles. The homotopy h is prescribed downstairs. At time $s = 0$, α tells us which point above $h(z, 0)$ to use for every z . The theorem says that these choices determine the entire motion upstairs, continuously and uniquely. It does not assert that every map $Z \rightarrow Y$ admits a lift: the initial lift is part of the hypothesis.

Corollary 3.9 (Path lifting). *If $\gamma : I \rightarrow Y$ is a path and $x \in X$ satisfies $f(x) = \gamma(0)$, then there is a unique path $\tilde{\gamma} : I \rightarrow X$ such that*

$$f \circ \tilde{\gamma} = \gamma, \quad \tilde{\gamma}(0) = x.$$

Proof. Take Z to be a one-point space in the homotopy lifting theorem. □

3.6 Degree of a loop in S^1

Return to the covering $f : \mathbb{R} \rightarrow S^1$, $f(x) = e^{2\pi ix}$. Let $\gamma : I \rightarrow S^1$ be a based loop at 1. Path lifting gives a unique $\tilde{\gamma} : I \rightarrow \mathbb{R}$ with

$$\tilde{\gamma}(0) = 0, \quad e^{2\pi i \tilde{\gamma}(t)} = \gamma(t).$$

Since $\gamma(1) = 1$, its endpoint lies in $f^{-1}(1) = \mathbb{Z}$.

Definition 3.10. The *degree*, or *winding number*, of γ is

$$\deg(\gamma) = \tilde{\gamma}(1) \in \mathbb{Z},$$

where the lift starts at 0.

The real coordinate records the total angle without reducing it modulo one turn. The constant loop has degree 0. The positive loop $t \mapsto e^{2\pi it}$ has lift $t \mapsto t$ and degree 1; the negative loop $t \mapsto e^{-2\pi it}$ has degree -1 .

For a loop with any starting point on the circle, choose a lift starting at any real preimage of that point and set

$$\deg(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0). \quad (3.6)$$

The difference is an integer, because the two endpoints have the same image under f . It is independent of the chosen starting lift: two possible starting points differ by an integer n , and path uniqueness says that the corresponding lifts differ by n everywhere. Subtracting endpoints removes that common integer. For a based loop at 1, this agrees with Definition 3.10.

3.7 Homotopy invariance of degree

Proposition 3.11. *Based homotopic loops in $(S^1, 1)$ have the same degree.*

Proof. Let $h : I \times I \rightarrow S^1$ be a based homotopy from γ_1 to γ_2 , so

$$h(t, 0) = \gamma_1(t), \quad h(t, 1) = \gamma_2(t), \quad h(0, s) = h(1, s) = 1.$$

Now view h as a homotopy in the t -direction, with s as the parameter. Apply homotopy lifting to $(s, t) \mapsto h(t, s)$, with $Z = I$ and the constant initial lift $\alpha(s) = 0$. Writing the lifted map in the original coordinate order gives a continuous map $\tilde{h} : I \times I \rightarrow \mathbb{R}$ satisfying

$$e^{2\pi i \tilde{h}(t, s)} = h(t, s), \quad \tilde{h}(0, s) = 0.$$

Since $h(1, s) = 1$, the function

$$s \mapsto \tilde{h}(1, s)$$

is continuous and takes values in $\mathbb{Z}$, hence is constant. For each s , the path $t \mapsto \tilde{h}(t, s)$ is the unique lift starting at 0. Therefore

$$\deg(\gamma_1) = \tilde{h}(1, 0) = \tilde{h}(1, 1) = \deg(\gamma_2). \quad \square$$

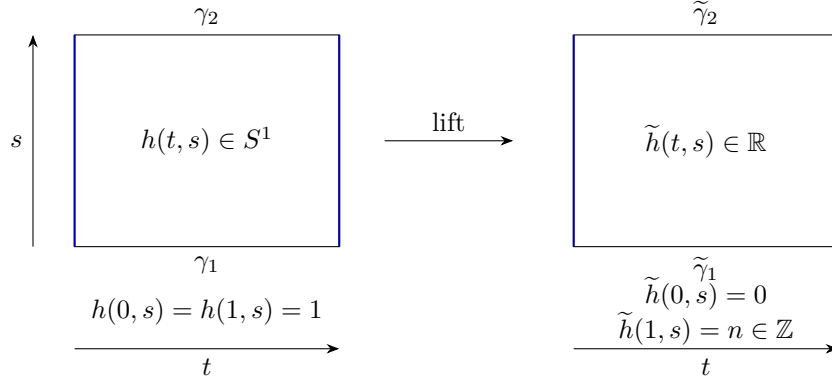


Figure 8: Lift in the t -direction from the constant initial lift on the left edge. The right edge is integer-valued and therefore constant.

The distinction in Figure 8 is essential: a loop has equal endpoint values, but its lift need not. The lifted endpoints can differ by a nonzero integer. For example, the loop $\gamma(t) = e^{2\pi it}$ lifts to $\tilde{\gamma}(t) = t$, whose endpoint values are 0 and 1.

We have therefore constructed a well-defined map of sets

$$\deg : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}, \quad [\gamma] \longmapsto \deg(\gamma).$$

This is the endpoint of the lecture. To complete the announced isomorphism $\pi_1(S^1, 1) \cong \mathbb{Z}$, it remains to show that this map respects the group operation and is bijective.

4 The circle, functoriality, and change of basepoint

Lecture 4, 14 October 2026

4.1 The fundamental group of the circle

In Lecture 3 we constructed the degree map

$$\deg : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}.$$

For a loop $\gamma : I \rightarrow S^1$ based at 1, its degree is the endpoint of the unique lift $\tilde{\gamma} : I \rightarrow \mathbb{R}$ starting at 0, under the covering $f(x) = e^{2\pi i x}$. Proposition 3.11 shows that this depends only on the based homotopy class of γ .

Theorem 4.1. *The degree map is an isomorphism of groups*

$$\deg : \pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}.$$

Proof. Surjectivity. For $n \in \mathbb{Z}$, consider

$$\gamma_n(t) = e^{2\pi i n t}, \quad t \in I.$$

The path $t \mapsto nt$ starts at 0 and satisfies $f(nt) = \gamma_n(t)$, so it is the required lift. Its endpoint is n ; hence $\deg(\gamma_n) = n$. The constant loop has degree 0.

Compatibility with multiplication. Let γ_1, γ_2 be based loops, and let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be their lifts starting at 0. Recall that

$$(\gamma_1\gamma_2)(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Its lift starting at 0 is

$$\widetilde{\gamma_1\gamma_2}(t) = \begin{cases} \tilde{\gamma}_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ \deg(\gamma_1) + \tilde{\gamma}_2(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

At $t = \frac{1}{2}$ both expressions equal $\deg(\gamma_1)$, so the pieces paste continuously. Adding the integer $\deg(\gamma_1)$ does not change the image under f , which verifies the lifting identity on the second half. Taking the endpoint gives

$$\deg(\gamma_1\gamma_2) = \deg(\gamma_1) + \deg(\gamma_2).$$

Thus $\deg$ is a group homomorphism.

Injectivity. It remains to show that its kernel is trivial. Suppose $\deg(\gamma) = 0$. Then the lift satisfies $\tilde{\gamma}(0) = \tilde{\gamma}(1) = 0$, so it is itself a based loop in $\mathbb{R}$. Contract it to 0 inside $\mathbb{R}$, keeping both endpoints at 0. Composing this contraction with f gives a based homotopy from γ to the constant loop at 1. Hence $[\gamma]$ is the identity, proving injectivity. $\square$

4.2 Application: the Brouwer fixed-point theorem

Write $D^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ for the closed unit disk. The following application was also the subject of Sheet 1, Problem 4.

Theorem 4.2 (Brouwer fixed-point theorem in dimension two). *Every continuous map $g : D^2 \rightarrow D^2$ has a fixed point.*

Proof sketch. First, there can be no retraction $r : D^2 \rightarrow S^1$. Indeed, the loop $\gamma(t) = e^{2\pi it}$ contracts to 1 inside the disk by

$$h(t, s) = (1 - s)\gamma(t) + s.$$

This fixes its endpoints. If r were a retraction, $r \circ h$ would contract the same loop inside S^1 , contradicting its degree 1.

Now suppose that g has no fixed point. For each $x \in D^2$, draw the ray from $g(x)$ through x , and let $r(x)$ be its intersection with S^1 beyond or at x . Since $g(x) \neq x$, this construction is well defined and varies continuously with x . If $x \in S^1$, then $r(x) = x$. Thus r would be a retraction, a contradiction. $\square$

4.3 Functors

In the proof of Theorem 4.1, composing with the covering map carried a contraction in $\mathbb{R}$ to a contraction in S^1 . More generally, maps between spaces induce maps between their algebraic invariants. The notion of a functor expresses the compatibility of these constructions with composition.

Definition 4.3. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ assigns an object $F(X)$ to every object X , and a morphism $F(f) : F(X) \rightarrow F(Y)$ to every morphism $f : X \rightarrow Y$, such that

$$F(\text{id}_X) = \text{id}_{F(X)}, \quad F(g \circ f) = F(g) \circ F(f)$$

whenever f, g are composable.

Lemma 4.4. A functor takes isomorphic objects to isomorphic objects.

Proof. If $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are mutually inverse, then

$$F(g) \circ F(f) = F(g \circ f) = \text{id}_{F(X)}, \quad F(f) \circ F(g) = F(f \circ g) = \text{id}_{F(Y)}.$$

Thus $F(f)$ and $F(g)$ are mutually inverse. $\square$

4.4 The fundamental group as a functor

Let $f : X \rightarrow Y$ be continuous with $f(x_0) = y_0$. A loop γ based at x_0 gives the loop $f \circ \gamma$ based at y_0 .

Proposition 4.5. Composition with f induces a group homomorphism

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \quad [\gamma] \longmapsto [f \circ \gamma].$$

Proof. If h is a based homotopy from γ_1 to γ_2 , then $f \circ h$ is a based homotopy from $f \circ \gamma_1$ to $f \circ \gamma_2$. Thus f_* is well defined. The identity

$$f \circ (\gamma_1 \gamma_2) = (f \circ \gamma_1)(f \circ \gamma_2)$$

shows that it preserves multiplication, and the constant loop maps to the constant loop. $\square$

Definition 4.6. The category **PTop** of *pointed topological spaces* has objects (X, x_0) , where X is a topological space and $x_0 \in X$. A morphism $(X, x_0) \rightarrow (Y, y_0)$ is a continuous map $f : X \rightarrow Y$ with $f(x_0) = y_0$. Identities and composition are those of continuous maps.

For composable pointed maps f, g , and for every loop γ ,

$$(g \circ f)_*([\gamma]) = [g \circ f \circ \gamma] = g_*(f_*([\gamma])).$$

Also $(\text{id}_X)_* = \text{id}_{\pi_1(X, x_0)}$. Consequently, the assignments $(X, x_0) \mapsto \pi_1(X, x_0)$ and $f \mapsto f_*$ define a functor

$$\pi_1 : \mathbf{PTop} \longrightarrow \mathbf{Groups}.$$

4.5 Changing the basepoint

We now compare fundamental groups at different basepoints.

Theorem 4.7. *If $x_1, x_2 \in X$ are joined by a path, then*

$$\pi_1(X, x_1) \cong \pi_1(X, x_2).$$

Warning. *Which isomorphism?* The theorem asserts that the groups are isomorphic, but does not single out an isomorphism. We will construct one by choosing a path, and different paths can give different isomorphisms. In Figure 10, the small loop γ around the rightmost puncture is evident at x_1 . To view it from x_2 , we must choose a route past the left puncture. The two choices δ, δ' give different images of $[\gamma]$, as we will see below.

Choose a path $\delta : I \rightarrow X$ from x_1 to x_2 . For a loop γ based at x_1 , define

$$\gamma_\delta(t) = \begin{cases} \delta(1 - 3t), & 0 \leq t \leq \frac{1}{3}, \\ \gamma(3t - 1), & \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \delta(3t - 2), & \frac{2}{3} \leq t \leq 1. \end{cases}$$

The formulas agree at $t = \frac{1}{3}, \frac{2}{3}$, where their value is x_1 , and both endpoints have value x_2 . Thus γ_δ is a continuous loop based at x_2 : it follows δ backwards, traverses γ , and returns along δ . We set

$$\beta_\delta : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_2), \quad \beta_\delta([\gamma]) = [\gamma_\delta]. \quad (4.1)$$

Proof of Theorem 4.7. Well-definedness. If h is a based homotopy from γ_1 to γ_2 at x_1 , replace the middle expression $\gamma(3t - 1)$ by $h(3t - 1, s)$ and leave the two outer thirds fixed. This gives a based homotopy from $(\gamma_1)_\delta$ to $(\gamma_2)_\delta$ at x_2 (Figure 9). The values agree at the seams, so continuity follows by pasting.

The homomorphism property. Put the loop $(\gamma_1)_\delta(\gamma_2)_\delta$ along the bottom of a square. Its six successive pieces occupy equal subintervals. The middle two follow δ forwards and then backwards. Contract this excursion using the triangular construction from Section 3.2, keeping the first two and last two pieces fixed (Figure 11, left). The vertical sides stay at x_2 . Removing the constant pause at the top and adjusting the speed gives $(\gamma_1\gamma_2)_\delta$. Consequently,

$$\beta_\delta([\gamma_1])\beta_\delta([\gamma_2]) = [(\gamma_1)_\delta(\gamma_2)_\delta] = [(\gamma_1\gamma_2)_\delta] = \beta_\delta([\gamma_1][\gamma_2]).$$

An inverse. Apply the same three-piece formula to the reversed path $\delta^{-1}(t) = \delta(1 - t)$. The loop $(\gamma_\delta)_{\delta^{-1}}$ first follows δ forwards and backwards, then traverses γ , and finally follows δ forwards and backwards once more. After adjusting the time intervals, put the triangular contraction in each outer strip and keep γ fixed in the middle (Figure 11, right). Both vertical sides stay at x_1 , and the top edge is γ with constant pauses at its ends. Hence

$$(\beta_{\delta^{-1}} \circ \beta_\delta)([\gamma]) = [(\gamma_\delta)_{\delta^{-1}}] = [\gamma].$$

The same construction with δ^{-1} in place of δ gives $\beta_\delta \circ \beta_{\delta^{-1}} = \text{id}$. Thus β_δ is an isomorphism, with inverse $\beta_{\delta^{-1}}$. $\square$

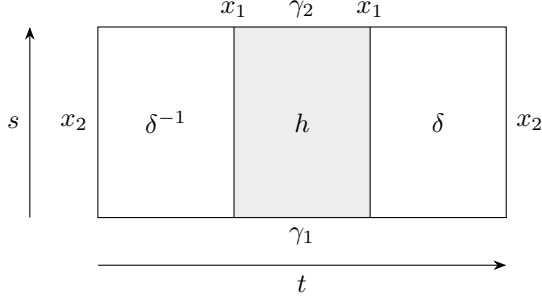


Figure 9: A based homotopy after changing the basepoint. The outer paths stay fixed, and the middle strip carries h . Each strip has width $\frac{1}{3}$, as in the defining formula.

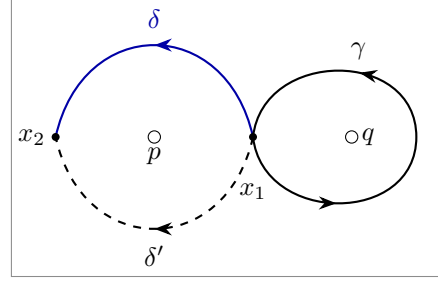


Figure 10: The loop γ is based at x_1 , beside the rightmost puncture. The two paths from x_1 to x_2 give different images of $[\gamma]$, and hence different isomorphisms.

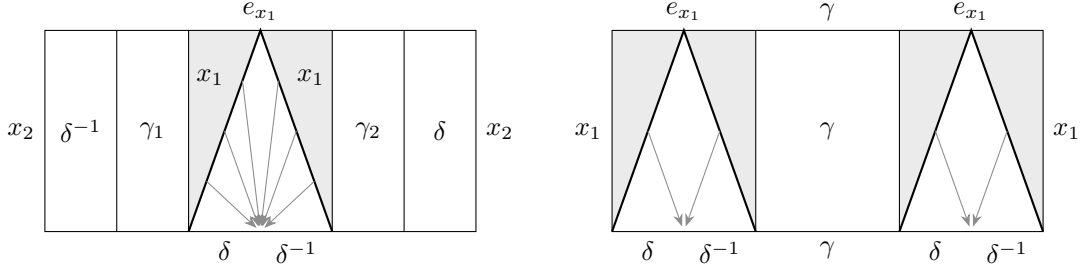


Figure 11: Cancellation squares, read from bottom to top. Left: the homomorphism property. Right: $\beta_{\delta^{-1}} \circ \beta_{\delta}$ is the identity, after rescaling the time intervals. The shaded triangles are constant at x_1 ; each fan carries δ towards its lower vertex.

4.6 Dependence on the connecting path

Let δ, δ' be two paths from x_1 to x_2 . Applying the three-piece formula twice gives the loop $(\gamma_{\delta})_{(\delta')^{-1}}$, representing $(\beta_{\delta'}^{-1} \circ \beta_{\delta})([\gamma])$. Its successive pieces follow δ' , then δ backwards, traverse γ , return along δ , and follow δ' backwards.

Define a loop u based at x_1 by

$$u(t) = \begin{cases} \delta(2t), & 0 \leq t \leq \frac{1}{2}, \\ \delta'(2-2t), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Draw the five pieces along the top of a rectangle, put γ on the bottom, and put u on both vertical sides, directed upwards (Figure 12).

To fill the rectangle at height s , start at $u(s)$, follow the initial segment of u backwards to x_1 , traverse γ , and return along the same initial segment to $u(s)$. The two outer pieces shrink to length zero at the bottom. The middle region carries γ , with a change of speed. Thus the basepoint of each horizontal loop is $u(s)$.

At the top, the first two pieces traverse u^{-1} , and the last two traverse u . All three of u^{-1}, γ, u are loops at x_1 , so their loop product is defined. Reading the top edge, up to a change of speed, gives

$$(\beta_{\delta'}^{-1} \circ \beta_{\delta})([\gamma]) = [u^{-1}\gamma u] = [u]^{-1}[\gamma][u].$$

The resulting automorphism of $\pi_1(X, x_1)$ is conjugation.

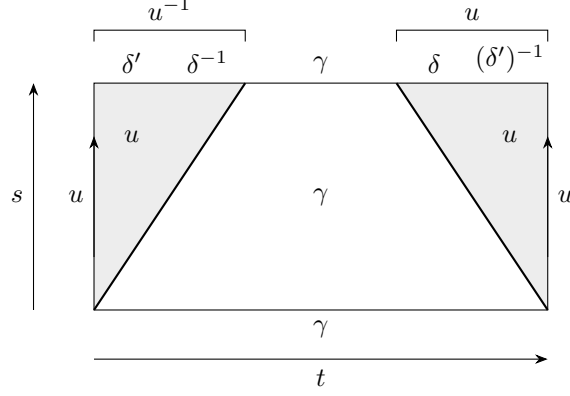


Figure 12: Complete the composite loop to a rectangle and fill it using γ and initial segments of u . The basepoint follows u along the vertical sides. The top edge reads $u^{-1}\gamma u$.

4.7 Conjugation and words in two generators

Let X be a rectangle with two punctures p, q , and choose a basepoint x_0 to their left. Let $a, b \in \pi_1(X, x_0)$ be represented by the loops drawn in Figure 13. The loop labelled a goes counterclockwise around p . The loop labelled b passes above p , goes counterclockwise around q , and returns along the same route. Outgoing and returning portions of a shared route are drawn slightly apart.

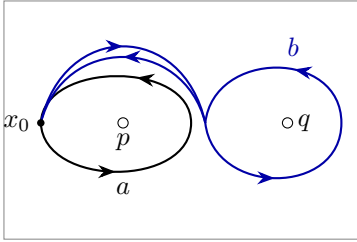
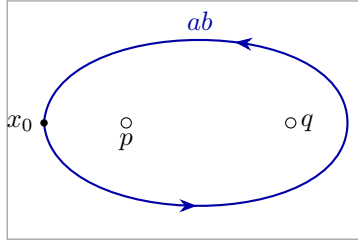
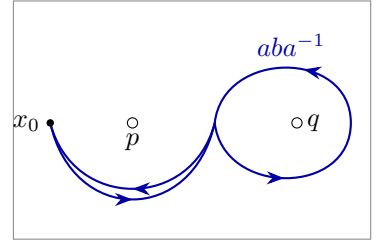


Figure 13: The generators a and b .

Figure 14: The product ab .

Figure 15: The conjugate aba^{-1} .

We will prove later that $\pi_1(X, x_0)$ is the *free group* generated by a and b . Its elements are reduced words in a, a^{-1}, b, b^{-1} : no letter is immediately followed by its inverse. Multiplication is concatenation followed by cancellation.

For example, ab is represented by following the loop labelled a , then the one labelled b . A based homotopy turns this into the single counterclockwise loop around both punctures in Figure 14. For aba^{-1} , follow these by the loop labelled a backwards. A based homotopy gives the representative in Figure 15, which reaches q below p . The reduced words b and aba^{-1} are distinct.

In any group, conjugation by a fixed element is an automorphism, called an *inner automorphism*. Here the inner automorphism $g \mapsto aga^{-1}$ takes b to aba^{-1} .

Returning to Figure 10, take $x_0 = x_2$. The upper path δ gives b , whereas the lower path δ' gives aba^{-1} :

$$\beta_\delta([\gamma]) = b, \quad \beta_{\delta'}([\gamma]) = aba^{-1}.$$

Thus the two connecting paths give different isomorphisms. Their difference is described by conjugation by a , giving a geometric interpretation of this inner automorphism.

5 Proof of the homotopy lifting theorem

Lecture 5, 15 October 2026

5.1 The lifting problem

We now prove Theorem 3.8. Let $f : X \rightarrow Y$ be a covering map and let

$$h : Z \times I \longrightarrow Y$$

be continuous. A *lift* of h is a continuous map $\tilde{h} : Z \times I \rightarrow X$ such that $f \circ \tilde{h} = h$. More generally, for a subset $A \subseteq Z \times I$, a *lift* of $h|_A$ is a continuous map $\tilde{h} : A \rightarrow X$ such that $f \circ \tilde{h} = h|_A$.

We are given an initial lift $\alpha : Z \rightarrow X$, satisfying

$$f(\alpha(z)) = h(z, 0) \quad (z \in Z).$$

We must construct a unique lift $\tilde{h}$ with $\tilde{h}(z, 0) = \alpha(z)$.

The proof has three parts: construct lifts on $U \times I$ for sufficiently small open neighborhoods U of each point of Z , prove that they agree on overlaps, and glue them. Uniqueness also enters the local construction, so we begin with two principles that will be used throughout.

5.2 Good sets and two lifting principles

We first express the covering condition in product form. Every point of Y has an open neighborhood V for which there are a discrete space Λ and a homeomorphism

$$\varphi : V \times \Lambda \xrightarrow{\cong} f^{-1}(V), \quad f(\varphi(y, \lambda)) = y.$$

Thus, in these coordinates, f is the projection onto V .

Definition 5.1. An open subset $V \subseteq Y$ with such a description is called *good*. A subset $A \subseteq Z \times I$ is called *good* if $h(A)$ is contained in a good subset of Y .

The inverse images under h of good open subsets of Y form an open cover of $Z \times I$ by good subsets.

Lemma 5.2 (Extension over a good set). *Let $A \subseteq Z \times I$ be good, let $p \in A$, and choose $x \in X$ with $f(x) = h(p)$. There is a lift of $h|_A$ whose value at p is x .*

Proof. Choose a good set V containing $h(A)$ and a homeomorphism $\varphi : V \times \Lambda \rightarrow f^{-1}(V)$ as above. There is a unique $\lambda \in \Lambda$ such that $x = \varphi(h(p), \lambda)$. The formula

$$\tilde{h}(q) = \varphi(h(q), \lambda), \quad q \in A,$$

defines the required continuous lift. □

Lemma 5.3 (Agreement of lifts). *Let $\tilde{h}_1, \tilde{h}_2 : A \rightarrow X$ be two lifts of $h|_A$, where $A \subseteq Z \times I$. Their agreement set*

$$E = \{p \in A : \tilde{h}_1(p) = \tilde{h}_2(p)\}$$

is both open and closed in A . In particular, if A is connected and the two lifts agree at one point, then they agree everywhere.

Proof. Fix $p \in A$ and choose a good neighborhood V of $h(p)$. On the relative open neighborhood $A \cap h^{-1}(V)$, write

$$\varphi^{-1}(\tilde{h}_j(q)) = (h(q), \lambda_j(q)), \quad j = 1, 2.$$

Each λ_j is continuous with values in the discrete space Λ , so both are constant on a sufficiently small neighborhood of p . If the two lifts agree at p , their λ -values agree on this neighborhood; if they differ at p , their λ -values remain different there. Thus both E and its complement are open. If A is connected and E is nonempty, it follows that $E = A$. □

5.3 A lift near $\{z\} \times I$

Lemma 5.4 (Tube lemma). *Let U be a topological space, let K be compact, and let $z \in U$. If an open subset $O \subseteq U \times K$ contains $\{z\} \times K$, then there is an open neighborhood W of z in U such that*

$$W \times K \subseteq O.$$

Proof. For each $k \in K$, choose open neighborhoods $W_k \subseteq U$ of z and $V_k \subseteq K$ of k such that $W_k \times V_k \subseteq O$. Compactness gives finitely many $k_1, \dots, k_m$ with $K = V_{k_1} \cup \dots \cup V_{k_m}$. Then

$$W = W_{k_1} \cap \dots \cap W_{k_m}$$

is an open neighborhood of z , and $W \times K \subseteq O$. $\square$

Fix $z \in Z$. We will find an open neighborhood U of z and a lift on $U \times I$ with initial value $\alpha|_U$.

Choosing finitely many good rectangles. For each $t \in I$, choose an open neighborhood U_t of z and an open subinterval J_t of I containing t such that

$$U_t \times \overline{J_t}$$

is good. This is possible by first choosing a product neighborhood inside a good open set and then taking a smaller interval whose closure stays inside that neighborhood. All closures here are taken in I . Open subintervals at the endpoints may have the forms $[0, b)$ and $(a, 1]$; the whole interval I is allowed as well.

By compactness of I , the open intervals J_t admit a finite subcover $J_1, \dots, J_n$, so $I = J_1 \cup \dots \cup J_n$. Write the corresponding good rectangles as

$$U_i \times \overline{J_i}, \quad i = 1, \dots, n.$$

Discard intervals contained in others and order the remaining ones by their left endpoints. Their right endpoints are then in increasing order as well. The first interval contains 0, the last contains 1, and consecutive intervals overlap.

Extending successive lifts. By Lemma 5.2, the prescribed value $\alpha(z)$ at $(z, 0)$ extends to a lift

$$\tilde{h}_1 : U_1 \times \overline{J_1} \longrightarrow X.$$

Having constructed $\tilde{h}_i$, choose $t_i \in J_i \cap J_{i+1}$ and apply the same lemma to $U_{i+1} \times \overline{J_{i+1}}$, prescribing the value $\tilde{h}_i(z, t_i)$ at (z, t_i) . This produces lifts $\tilde{h}_i$ on all the rectangles.

Making the lifts agree on overlaps. Set

$$K_i = \overline{J_i} \cap \overline{J_{i+1}}, \quad i = 1, \dots, n-1.$$

The interval K_i is connected, and $\tilde{h}_i$ and $\tilde{h}_{i+1}$ agree at (z, t_i) . Lemma 5.3 therefore shows that they agree on $\{z\} \times K_i$.

Their agreement set is also open in $(U_i \cap U_{i+1}) \times K_i$. Since K_i is compact, Lemma 5.4 gives an open neighborhood $W_i \subseteq U_i \cap U_{i+1}$ of z such that they agree on

$$W_i \times K_i.$$

This is the reason for using the closures of the intervals: the overlap used in this shrinking step is compact.

We must also retain the initial condition. On $U_1 \times \{0\}$, the two lifts $(u, 0) \mapsto \tilde{h}_1(u, 0)$ and $(u, 0) \mapsto \alpha(u)$ agree at $(z, 0)$. Their agreement set is open, so there is an open neighborhood $U_0 \subseteq U_1$ of z on which

$$\tilde{h}_1(u, 0) = \alpha(u).$$

Take the intersection of all the finitely many neighborhoods:

$$U = U_0 \cap \bigcap_{i=1}^n U_i \cap \bigcap_{i=1}^{n-1} W_i.$$

On $U \times J_i$, the restricted lifts agree for neighboring indices. They agree on every pairwise overlap as well: if $i < j < k$, the ordering of the interval endpoints gives

$$J_i \cap J_k \subseteq J_j.$$

At a point belonging to $J_i \cap J_k$, agreement therefore passes through all the intermediate lifts.

The sets $U \times J_i$ form an open cover of $U \times I$. By Lemma 5.5, the lifts on them glue to a continuous lift

$$\tilde{h}_U : U \times I \longrightarrow X, \quad \tilde{h}_U(u, 0) = \alpha(u).$$

This completes the local construction.

5.4 Uniqueness and global gluing

For each $z \in Z$, the interval $\{z\} \times I$ is connected. Consequently, two lifts on this interval with the same value at $(z, 0)$ agree everywhere, by Lemma 5.3.

The local construction gives an open cover $\{U_\sigma\}$ of Z and lifts

$$\tilde{h}_\sigma : U_\sigma \times I \longrightarrow X$$

with the prescribed initial values. If $z \in U_\sigma \cap U_\tau$, both lifts restrict to lifts on $\{z\} \times I$ starting at $\alpha(z)$, so they agree there. Thus the local lifts agree on every overlap $(U_\sigma \cap U_\tau) \times I$.

Recall the gluing principle used already in the local construction.

Lemma 5.5 (Gluing over an open cover). *Let $\{A_\sigma\}$ be an open cover of a space A . If continuous maps $g_\sigma : A_\sigma \rightarrow B$ agree on every overlap, then there is a unique continuous map $g : A \rightarrow B$ whose restriction to each A_σ is g_σ .*

Proof. For $a \in A$, choose σ with $a \in A_\sigma$ and set $g(a) = g_\sigma(a)$. Agreement on overlaps makes this well-defined and also gives uniqueness. For an open set $O \subseteq B$,

$$g^{-1}(O) = \bigcup_{\sigma} g_\sigma^{-1}(O).$$

Each set in this union is open in A_σ , hence in A . Therefore g is continuous. □

Applying Lemma 5.5 to the open cover $\{U_\sigma \times I\}$ gives a continuous map

$$\tilde{h} : Z \times I \longrightarrow X, \quad f \circ \tilde{h} = h, \quad \tilde{h}(z, 0) = \alpha(z).$$

Finally, any two such global lifts agree on every $\{z\} \times I$ by the same connectedness argument. Hence the lift is unique, proving Theorem 3.8.

6 Coverings and monodromy

Lecture 6, 21 October 2026

The covering $\mathbb{R} \rightarrow S^1$ allowed us to study loops in the circle by lifting them to the real line. We now apply the same idea to an arbitrary covering. The endpoints of lifted loops will describe an action of the fundamental group on a fibre.

6.1 More examples of coverings

Example 6.1 (Products and the torus). The product of two covering maps $f_i : X_i \rightarrow Y_i$, $i = 1, 2$, is a covering map

$$f_1 \times f_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2.$$

Indeed, if $U_i \subseteq Y_i$ is evenly covered, with sheets $V_{i,\alpha}$, then the inverse image of $U_1 \times U_2$ is the disjoint union of the open sets $V_{1,\alpha} \times V_{2,\beta}$. Each maps homeomorphically onto $U_1 \times U_2$.

In particular, we have the covering

$$\mathbb{R}^2 \longrightarrow S^1 \times S^1, \quad (u, v) \longmapsto (e^{2\pi i u}, e^{2\pi i v}).$$

The product $S^1 \times S^1$ is the *torus*, denoted T^2 . One circle records the position around the central hole, and the other records the position around the circular cross-section.

Definition 6.2 (Real projective space). The *real projective space* $\mathbb{RP}^n$ is the space of one-dimensional vector subspaces of $\mathbb{R}^{n+1}$, that is, lines through the origin. Equivalently,

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim, \quad v \sim \lambda v \quad (\lambda \in \mathbb{R} \setminus \{0\}).$$

Each line meets S^n in exactly two antipodal points, so we also have the description

$$\mathbb{RP}^n = S^n / (v \sim -v).$$

We use the quotient topology: a subset is open precisely when its inverse image under the quotient map is open.

The projection

$$q : S^n \longrightarrow \mathbb{RP}^n, \quad v \longmapsto [v],$$

is a double covering. To see this, choose a small open cap V around v disjoint from its antipodal cap $-V$. Then $q(V)$ is open, its inverse image is $V \sqcup (-V)$, and each cap maps homeomorphically onto $q(V)$.

For $n = 1$, a line in $\mathbb{R}^2$ is determined by an angle modulo π , rather than modulo 2π . Thus $\mathbb{RP}^1$ is a circle. Under the identification $[e^{i\theta}] \mapsto e^{2i\theta}$, the covering $S^1 \rightarrow \mathbb{RP}^1$ becomes the familiar double covering $z \mapsto z^2$.

For $n = 2$, split the sphere into its two open hemispheres and the equator. The antipodal identification pairs the two hemispheres. We may therefore keep one closed hemisphere, which is a disk, and identify opposite points of its boundary. This gives a disk model of $\mathbb{RP}^2$.

Another description is the plane together with points at infinity. Lines represented by $(x, y, 1)$ form a copy of $\mathbb{R}^2$; the remaining lines, represented by $(x, y, 0)$, form a copy of $\mathbb{RP}^1$ at infinity. Travelling out towards infinity along a straight line and continuing through the added point brings us back from the opposite direction.

This also gives an intuitive picture of nonorientability. Imagine a car with its steering wheel on the left. After a trip through the antipodal boundary identification and back to its starting place, left and right have been exchanged: the wheel is now on the right. Thus a choice of orientation cannot be made consistently throughout $\mathbb{RP}^2$.

6.2 The fibre and endpoints of lifted loops

Let $f : X \rightarrow Y$ be a covering map and fix $y_0 \in Y$. The set

$$S = f^{-1}(y_0)$$

is called the *fibre* over y_0 . It is discrete: an open sheet over an evenly covered neighborhood of y_0 meets S in a single point.

Let $\gamma : I \rightarrow Y$ be a loop based at y_0 . For each $s \in S$, path lifting (Corollary 3.9) gives a unique path $\tilde{\gamma}_s : I \rightarrow X$ satisfying

$$f \circ \tilde{\gamma}_s = \gamma, \quad \tilde{\gamma}_s(0) = s.$$

Since $\gamma(1) = y_0$, its endpoint again belongs to S . Define

$$a_\gamma : S \longrightarrow S, \quad a_\gamma(s) = \tilde{\gamma}_s(1).$$

Thus a_γ records where each point of the fibre goes when we lift γ starting at that point. A lifted loop need not be a loop upstairs: its endpoint may differ from its starting point.

Proposition 6.3. *If γ_1 and γ_2 are based homotopic loops in (Y, y_0) , then $a_{\gamma_1} = a_{\gamma_2}$.*

Proof. Let $h : I \times I \rightarrow Y$ be a based homotopy, so that

$$h(t, 0) = \gamma_1(t), \quad h(t, 1) = \gamma_2(t), \quad h(0, u) = h(1, u) = y_0.$$

Fix $s \in S$. View h as a homotopy in the t -direction, with u as the parameter. Apply the homotopy lifting theorem to $(u, t) \mapsto h(t, u)$, with constant initial lift $u \mapsto s$. Writing the lift in the original coordinate order gives

$$\tilde{h} : I \times I \longrightarrow X, \quad f \circ \tilde{h} = h, \quad \tilde{h}(0, u) = s.$$

For each u , the path $t \mapsto \tilde{h}(t, u)$ is the unique lift starting at s of the loop $t \mapsto h(t, u)$. The endpoint map

$$I \longrightarrow S, \quad u \longmapsto \tilde{h}(1, u)$$

is continuous. Since I is connected and S is discrete, this map is constant. Consequently,

$$a_{\gamma_1}(s) = \tilde{h}(1, 0) = \tilde{h}(1, 1) = a_{\gamma_2}(s).$$

This holds for every $s \in S$.

Notice also what the lifted homotopy tells us when $a_{\gamma_1}(s) = s$. In this case

$$\tilde{h}(0, u) = \tilde{h}(1, u) = s \quad \text{for every } u \in I, \tag{6.1}$$

so $\tilde{h}$ is a based homotopy between the two lifted loops in (X, s) . $\square$

This is the same argument as the homotopy invariance of degree in Proposition 3.11. There the fibre was $\mathbb{Z} \subset \mathbb{R}$; here it is an arbitrary discrete set S . We may therefore write $a_{[\gamma]}$ for a_γ .

6.3 Right and left actions

Recall that our product $\gamma\delta$ means: first traverse γ , then traverse δ . This order determines the action convention.

Definition 6.4. Let G be a group with neutral element e and S a set. A *covariant*, or *left*, action assigns a bijection $B_g : S \rightarrow S$ to each $g \in G$ such that

$$B_e = \text{id}_S, \quad B_{g_1 g_2} = B_{g_1} \circ B_{g_2}.$$

Writing $g \cdot s = B_g(s)$, this says $(g_1 g_2) \cdot s = g_1 \cdot (g_2 \cdot s)$. A *contravariant*, or *right*, action instead satisfies

$$B_e = \text{id}_S, \quad B_{g_1 g_2} = B_{g_2} \circ B_{g_1}.$$

Writing $s \cdot g = B_g(s)$, this says $s \cdot (g_1 g_2) = (s \cdot g_1) \cdot g_2$.

Proposition 6.5. The maps a_γ define a contravariant action of $\pi_1(Y, y_0)$ on S . In particular,

$$a_{\gamma\delta} = a_\delta \circ a_\gamma, \quad a_e = \text{id}_S, \quad a_\gamma^{-1} = a_{\gamma^{-1}},$$

where e is the constant loop at y_0 .

Proof. Fix $s \in S$ and put $s' = a_\gamma(s)$. Lift γ starting at s and δ starting at s' . The formula

$$\lambda(t) = \begin{cases} \tilde{\gamma}_s(2t), & 0 \leq t \leq \frac{1}{2}, \\ \tilde{\delta}_{s'}(2t - 1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

defines a continuous path: the two values at $t = \frac{1}{2}$ are both s' . It starts at s and projects to $\gamma\delta$, so uniqueness of path lifting identifies it with the lift of $\gamma\delta$ starting at s . Its endpoint is $a_\delta(a_\gamma(s))$, proving the product formula.

The lift of e starting at s is the constant path at s , so $a_e = \text{id}_S$. Since $\gamma\gamma^{-1}$ and $\gamma^{-1}\gamma$ are based homotopic to e , the product formula and Proposition 6.3 give

$$a_{\gamma^{-1}} \circ a_\gamma = \text{id}_S, \quad a_\gamma \circ a_{\gamma^{-1}} = \text{id}_S.$$

Thus a_γ is a bijection with inverse $a_{\gamma^{-1}}$. □

The product formula is the analogue of additivity of degree: lifting the first loop determines the starting point for lifting the second.

There are two ways to obtain a covariant action. One could reverse the convention for multiplication of loops. We keep our convention and instead put

$$a'_\gamma = a_\gamma^{-1} = a_{\gamma^{-1}}.$$

Then

$$a'_{\gamma\delta} = (a_\delta \circ a_\gamma)^{-1} = a_\gamma^{-1} \circ a_\delta^{-1} = a'_\gamma \circ a'_\delta.$$

Writing $\text{Bij}(S)$ for the group of bijections of S under composition, we obtain a homomorphism

$$\text{act} : \pi_1(Y, y_0) \longrightarrow \text{Bij}(S), \quad \text{act}([\gamma]) = a'_\gamma.$$

We call this the *monodromy action*. Thus $\text{act}([\gamma])(s)$ is the endpoint of the lift of γ^{-1} starting at s .

6.4 Examples of monodromy

Example 6.6 (The real line covering the circle). For $f : \mathbb{R} \rightarrow S^1$, $f(x) = e^{2\pi i x}$, the fibre over 1 is $S = \mathbb{Z}$. If $\deg(\gamma) = n$, the lift starting at $k \in \mathbb{Z}$ is the lift starting at 0 translated by k . Hence

$$a_\gamma(k) = k + n, \quad \text{act}(n)(k) = k - n,$$

where we identify $\pi_1(S^1, 1)$ with $\mathbb{Z}$ by degree. The minus sign comes from passing from a_γ to a'_γ .

Example 6.7 (The sphere covering the projective plane). For $S^2 \rightarrow \mathbb{RP}^2$, each fibre has two points, v and $-v$. A path in S^2 from v to $-v$ projects to a loop in $\mathbb{RP}^2$ whose monodromy interchanges them. We will prove later that $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$. Its nonidentity element acts by this interchange.

Example 6.8 (The double covering of the circle). For $S^1 \rightarrow S^1$, $z \mapsto z^2$, the fibre over 1 is $\{1, -1\}$. Under $\pi_1(S^1, 1) \cong \mathbb{Z}$, every even integer acts as the identity, and every odd integer interchanges the two points. Passing to the inverse permutation makes no difference to this action.

Example 6.9 (A trivial covering). For any space Y , the projection

$$Y \times \{1, 2\} \longrightarrow Y$$

is a covering. A lift of a loop stays in the copy of Y where it starts, and returns to its initial point. Thus the action on the two-point fibre is trivial.

Example 6.10 (Three circles and a five-point fibre). Let X be a disjoint union of three circles and define $f : X \rightarrow S^1$ by the double covering on each of the first two circles and the identity on the third. The fibre has five points. Label the two points in the first circle by 1, 2, those in the second by 3, 4, and the point in the third by 5. The generator $1 \in \mathbb{Z} \cong \pi_1(S^1, 1)$ acts as

$$(1\ 2)(3\ 4),$$

fixing 5. Every even integer acts trivially, and every odd integer acts by this same permutation.

6.5 Fundamental groups and stabilizers

Choose a point $s \in S$. The covering gives a homomorphism

$$f_* : \pi_1(X, s) \longrightarrow \pi_1(Y, y_0), \quad [\eta] \longmapsto [f \circ \eta].$$

Proposition 6.11. *The homomorphism f_* is injective.*

Proof. Let η be a loop in (X, s) such that $f_*([\eta]) = [e]$. Thus $f \circ \eta$ is based homotopic to the constant loop e in (Y, y_0) .

Recall the lifted homotopy constructed in the proof of Proposition 6.3. Apply that construction to this based homotopy, with starting point s . By uniqueness of path lifting, its first path is η and its last path is the constant path at s . Since η ends at s , equation (6.1) shows that the whole lifted homotopy is based at s . It therefore contracts η through based loops in X . Hence $[\eta] = [e]$, so $\ker f_*$ is trivial. $\square$

Definition 6.12. For a left action of a group G on a set S , the *stabilizer* of $s \in S$ is the subgroup

$$\text{Stab}(s) = \{g \in G : g \cdot s = s\}.$$

Proposition 6.13. *For the monodromy action,*

$$\text{im } f_* = \text{Stab}(s).$$

Consequently, f_ identifies $\pi_1(X, s)$ with the stabilizer of s in $\pi_1(Y, y_0)$.*

Proof. A bijection fixes s if and only if its inverse fixes s . Thus

$$\text{act}([\gamma])(s) = s \iff a_\gamma(s) = s \iff \tilde{\gamma}_s(1) = s.$$

If $[\gamma]$ stabilizes s , its lift $\tilde{\gamma}_s$ is a loop in (X, s) , and $f_*([\tilde{\gamma}_s]) = [\gamma]$.

Conversely, if η is a loop in (X, s) , then η is the lift of $f \circ \eta$ starting at s . Its endpoint is s , so $f_*([\eta])$ belongs to the stabilizer. $\square$

6.6 Orbits and path components

Definition 6.14. For a left action of G on S , the *orbit* of $s \in S$ is

$$G \cdot s = \{g \cdot s : g \in G\}.$$

The action is *transitive* if, for any $s_1, s_2 \in S$, there is a $g \in G$ with $g \cdot s_1 = s_2$.

Theorem 6.15. Suppose Y is path-connected. The monodromy orbits in $S = f^{-1}(y_0)$ are in bijection with the path components of X . The orbit of $s \in S$ is exactly the intersection of S with the path component containing s .

Proof. First, every path component of X meets S . Given $x \in X$, choose a path in Y from $f(x)$ to y_0 . Its lift starting at x ends at a point of S , in the same path component as x .

Now let $s_1, s_2 \in S$. If they lie in the same monodromy orbit, there is a loop γ with $\text{act}([\gamma])(s_1) = s_2$. The lift of γ^{-1} starting at s_1 is a path from s_1 to s_2 in X .

Conversely, suppose $\eta : I \rightarrow X$ is a path from s_1 to s_2 . Its projection $\gamma = f \circ \eta$ is a loop at y_0 . By uniqueness of path lifting, $\tilde{\gamma}_{s_1} = \eta$, so $a_\gamma(s_1) = s_2$. Equivalently, $\text{act}([\gamma]^{-1})(s_1) = s_2$. Thus s_1 and s_2 lie in the same orbit. $\square$

Corollary 6.16. If Y is path-connected, then X is path-connected if and only if the monodromy action is transitive.

In Example 6.10, the orbits are

$$\{1, 2\}, \quad \{3, 4\}, \quad \{5\}.$$

They correspond precisely to the three circle components of X .

7 Monodromy as a functor and the universal cover

Lecture 7, 22 October 2026

The monodromy action records the path components of a covering space and its fundamental groups. We now aim at a stronger statement: the action determines the covering itself, up to isomorphism. To formulate this precisely, we use categories of coverings and of group actions.

7.1 Two categories

Fix a space Y , a point $y_0 \in Y$, and write $G = \pi_1(Y, y_0)$.

Definition 7.1. The category $\text{Cov}(Y)$ has coverings $f : X \rightarrow Y$ as objects. A morphism from $f_1 : X_1 \rightarrow Y$ to $f_2 : X_2 \rightarrow Y$ is a continuous map $\alpha : X_1 \rightarrow X_2$ such that

$$f_2 \circ \alpha = f_1.$$

Composition and identities are those of continuous maps.

Thus a morphism is a map between the covering spaces which leaves their images in the base unchanged:

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha} & X_2 \\ & \searrow f_1 & \swarrow f_2 \\ & Y & \end{array}$$

An isomorphism in $\text{Cov}(Y)$ is a homeomorphism between the covering spaces commuting with their projections.

Definition 7.2. The category $G\text{-Set}$ has sets with a left G -action as objects. A morphism from S_1 to S_2 is a G -equivariant map: a map $u : S_1 \rightarrow S_2$ satisfying

$$u(g \cdot s) = g \cdot u(s) \quad (g \in G, s \in S_1).$$

Composition and identities are those of maps of sets.

The composite of two equivariant maps is equivariant, and identity maps are equivariant. An isomorphism in this category is an equivariant bijection.

7.2 The monodromy functor

To construct a functor $\Phi : \text{Cov}(Y) \rightarrow G\text{-Set}$, we must assign a G -set to every covering and an equivariant map to every morphism of coverings, with the corresponding source and target. We must then prove that these assignments preserve identity morphisms and composition.

Proposition 7.3. *The following assignments define a functor*

$$\Phi : \text{Cov}(Y) \longrightarrow G\text{-Set}.$$

1. On objects, assign to a covering $f : X \rightarrow Y$ the fibre $S_f = f^{-1}(y_0)$ with the covariant monodromy action

$$\Phi(f) = (S_f, \text{act}_f), \quad \text{act}_f([\gamma]) = a'_\gamma = a_{\gamma^{-1}}.$$

Here $a_\gamma(s)$ is the endpoint of the lift of γ starting at s .

2. On morphisms, assign to $\alpha : (f_1 : X_1 \rightarrow Y) \rightarrow (f_2 : X_2 \rightarrow Y)$ the restriction

$$\Phi(\alpha) : S_{f_1} \longrightarrow S_{f_2}, \quad s \longmapsto \alpha(s).$$

Proof. We check each requirement.

Objects. Section 6.3 shows that act_f is well-defined on $G = \pi_1(Y, y_0)$ and is a group homomorphism $G \rightarrow \text{Bij}(S_f)$. Thus it is a left G -action, so $\Phi(f)$ is indeed an object of $G\text{-Set}$.

Morphisms. Write $S_i = f_i^{-1}(y_0)$. If $s \in S_1$, then $f_2(\alpha(s)) = f_1(s) = y_0$, so $\alpha(s) \in S_2$. Thus the proposed map $\Phi(\alpha)$ has the required source and target. To be a morphism in $G\text{-Set}$, it must also satisfy

$$\Phi(\alpha)(g \cdot s) = g \cdot \Phi(\alpha)(s) \quad (g \in G, s \in S_1).$$

Let γ be a based loop in Y , and let $\tilde{\gamma}_s$ be its lift to X_1 starting at s . Since $f_2 \circ \alpha = f_1$, the path $\alpha \circ \tilde{\gamma}_s$ is a lift of γ to X_2 starting at $\alpha(s)$. Uniqueness of path lifting therefore gives, on taking endpoints,

$$\alpha(a_\gamma^{(1)}(s)) = a_\gamma^{(2)}(\alpha(s)).$$

Apply this to γ^{-1} . Since $[\gamma] \cdot s = a_{\gamma^{-1}}(s)$, it gives precisely the required equivariance.

Identities and composition. For every covering $f : X \rightarrow Y$, its identity morphism is id_X , whose restriction to S_f is id_{S_f} . Hence

$$\Phi(\text{id}_f) = \text{id}_{\Phi(f)}.$$

For composable morphisms of coverings $\alpha : f_1 \rightarrow f_2$ and $\beta : f_2 \rightarrow f_3$, every $s \in S_1$ satisfies

$$\Phi(\beta \circ \alpha)(s) = \beta(\alpha(s)) = \Phi(\beta)(\Phi(\alpha)(s)).$$

Therefore $\Phi(\beta \circ \alpha) = \Phi(\beta) \circ \Phi(\alpha)$. These are the two functor axioms. $\square$

The intended classification says that this functor has an inverse in an appropriate categorical sense. We first explain what that means.

7.3 Natural transformations and equivalence

When two constructions produce isomorphic objects, their outputs need not be literally equal. For example, the sets $\{a, b, c\}$ and $\{A, B, C\}$ are different sets, but the map $a \mapsto A$, $b \mapsto B$, $c \mapsto C$ is a bijection. For functors, we want isomorphisms between the outputs which are compatible with all morphisms.

Definition 7.4. Let $F_1, F_2 : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be functors. A *natural transformation* $\theta : F_1 \Rightarrow F_2$ assigns to each object $X \in \mathcal{C}_1$ a morphism

$$\theta_X : F_1(X) \longrightarrow F_2(X)$$

such that, for every morphism $\alpha : X_1 \rightarrow X_2$,

$$F_2(\alpha) \circ \theta_{X_1} = \theta_{X_2} \circ F_1(\alpha). \quad (7.1)$$

The condition says that the following square commutes:

$$\begin{array}{ccc} F_1(X_1) & \xrightarrow{\theta_{X_1}} & F_2(X_1) \\ F_1(\alpha) \downarrow & & \downarrow F_2(\alpha) \\ F_1(X_2) & \xrightarrow{\theta_{X_2}} & F_2(X_2) \end{array}$$

In particular, when $\alpha : X \rightarrow X$ is a symmetry of X , the comparison θ_X must intertwine the corresponding symmetries $F_1(\alpha)$ and $F_2(\alpha)$. Naturality is a compatibility requirement, not merely a choice of a map for each object.

Lemma 7.5. *If every component θ_X of a natural transformation is an isomorphism, then the maps θ_X^{-1} form a natural transformation $F_2 \Rightarrow F_1$.*

Proof. Composing equation (7.1) on the left with $\theta_{X_2}^{-1}$ and on the right with $\theta_{X_1}^{-1}$ gives

$$\theta_{X_2}^{-1} \circ F_2(\alpha) = F_1(\alpha) \circ \theta_{X_1}^{-1}.$$

This is precisely the naturality condition for the inverse components. □

Definition 7.6. A natural transformation whose components are all isomorphisms is a *natural isomorphism*. Functors $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ and $\Psi : \mathcal{D} \rightarrow \mathcal{C}$ are *mutually inverse up to natural isomorphism* if

$$\Psi \circ \Phi \cong \text{id}_{\mathcal{C}}, \quad \Phi \circ \Psi \cong \text{id}_{\mathcal{D}}$$

by natural isomorphisms. Such functors define an *equivalence of categories*.

For the classification of coverings, we work with a path-connected base Y which has a basis of simply connected open neighborhoods. In particular, these neighborhoods are path-connected. This is the local condition used in the construction below.

Theorem 7.7 (Classification of coverings, to be proved). *Under these hypotheses on Y , the monodromy functor*

$$\Phi : \text{Cov}(Y) \longrightarrow \pi_1(Y, y_0)\text{-Set}$$

is an equivalence of categories.

In particular, the theorem will give a bijection between isomorphism classes of coverings and isomorphism classes of G -sets. We begin the construction of an inverse functor by considering one particular G -set.

7.4 The regular action and universal coverings

The group G acts on itself by left translations:

$$g \cdot x = gx.$$

This action is transitive, and the stabilizer of every point is trivial. If a covering has this monodromy action, then Theorem 6.15 says that its covering space is path-connected, and Proposition 6.13 says that its fundamental group is trivial. Thus the covering space is simply connected.

Definition 7.8. A covering $f : \tilde{Y} \rightarrow Y$ is a *universal covering* of Y if $\tilde{Y}$ is simply connected.

We will construct such a covering directly. Its fibre will be identified with G , and we will recover simple connectedness from the monodromy action and its stabilizers.

7.5 The set of points of the universal cover

For $y \in Y$, let $P(y_0, y)$ denote the set of homotopy classes of paths from y_0 to y , keeping both endpoints fixed. Explicitly, paths $\gamma_1, \gamma_2 : I \rightarrow Y$ with these endpoints represent the same class if there is a continuous map $h : I \times I \rightarrow Y$ such that

$$h(t, 0) = \gamma_1(t), \quad h(t, 1) = \gamma_2(t), \quad h(0, u) = y_0, \quad h(1, u) = y.$$

Define the set

$$\tilde{Y} = \{(y, [\gamma]) : y \in Y, [\gamma] \in P(y_0, y)\}$$

and its projection

$$f : \tilde{Y} \longrightarrow Y, \quad f(y, [\gamma]) = y.$$

The constant path e at y_0 determines the distinguished point $\tilde{y}_0 = (y_0, [e])$.

Think of *Ariadne's thread*. You start at y_0 and walk through Y , laying a thread behind you that leads back to your starting point. A point of $\tilde{Y}$ is the state of this system. It records your current position y and the thread γ , considered up to continuous deformation with both ends fixed. This is the pair $(y, [\gamma])$.

Returning to the same position in Y need not restore the same state: the thread may have a different homotopy class. The projection f forgets the thread and records only your position. We next put a topology on this set.

7.6 Local sheets and the topology

Let $U \subseteq Y$ be a simply connected open neighborhood, and fix $y \in U$. For every $z \in U$, there is a path $\eta : I \rightarrow U$ from y to z , unique up to homotopy keeping its endpoints fixed. Given a path γ from y_0 to y , extend it to a path ending at z by the explicit formula

$$c_{\gamma, \eta}(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq \frac{1}{2}, \\ \eta(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The two formulas agree at $t = \frac{1}{2}$, and the pasting lemma gives continuity. Its homotopy class depends only on $[\gamma]$ and z : changing γ by a homotopy with fixed endpoints, or changing η through paths in U with the same endpoints, gives such a homotopy of the displayed path.

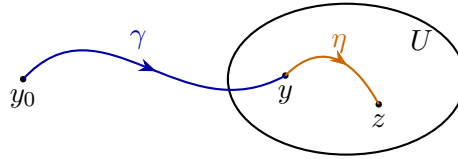


Figure 16: Extend a path to y by a path inside U . Simple connectedness of U makes the resulting class independent of the choice of the path from y to z .

For each $z \in U$, this construction gives a bijection $P(y_0, y) \rightarrow P(y_0, z)$. Its inverse extends a path ending at z by a path in U from z back to y . Going there and back adds a path in U homotopic, with fixed endpoints, to the constant path. We therefore obtain a bijection

$$\varphi_{U, y} : U \times P(y_0, y) \longrightarrow f^{-1}(U), \quad (z, [\gamma]) \longmapsto (z, [c_{\gamma, \eta}]). \quad (7.2)$$

For $[\gamma] \in P(y_0, y)$, write

$$\tilde{U}_{[\gamma]} = \varphi_{U, y}(U \times \{[\gamma]\}).$$

Declare these sets, for all such U , y , and $[\gamma]$, to be basic open subsets of $\tilde{Y}$.

Proposition 7.9. *These sets form a basis for a topology on $\tilde{Y}$. With this topology, $f : \tilde{Y} \rightarrow Y$ is a covering map, and equation (7.2) is a homeomorphism when $P(y_0, y)$ is given the discrete topology.*

Proof. Every point $(z, [\beta])$ lies in one of the proposed basic sets: choose a simply connected open neighborhood of z and use $[\beta]$ to specify its sheet.

Suppose a point $(z, [\beta])$ belongs to sheets over U_1 and U_2 . Choose a simply connected open neighborhood W of z contained in $U_1 \cap U_2$. The sheet over W specified by $[\beta]$ is contained in both original sheets. Indeed, extending from z along paths in W agrees with extending inside either U_i , by uniqueness of path classes in U_i . This verifies the intersection condition for a basis.

The same argument shows that f maps each sheet $\tilde{U}_{[\gamma]}$ homeomorphically onto U : it is a bijection by construction, and smaller sheets over simply connected open subsets give the corresponding neighborhood bases on both sides. The sheets over U are disjoint and exhaust $f^{-1}(U)$ by the bijection (7.2). Thus U is evenly covered, proving the covering property and the asserted product description. $\square$

7.7 Paths in the cover and its monodromy

We can describe a lifted path explicitly. Let $\gamma : I \rightarrow Y$ start at y_0 , and choose $x = (y_0, [\gamma_x]) \in f^{-1}(y_0)$, where γ_x is a loop at y_0 . For $t \in I$, define a path $\beta_t : I \rightarrow Y$ by

$$\beta_t(s) = \begin{cases} \gamma_x\left(\frac{s}{1-t/2}\right), & 0 \leq s \leq 1 - \frac{t}{2}, \\ \gamma\left(2\left(s - 1 + \frac{t}{2}\right)\right), & 1 - \frac{t}{2} \leq s \leq 1. \end{cases} \quad (7.3)$$

It first traverses γ_x , then the part of γ from time 0 to time t . The joining value is y_0 , and $\beta_t(1) = \gamma(t)$. At $t = 0$, the second piece has length zero and $\beta_0 = \gamma_x$.

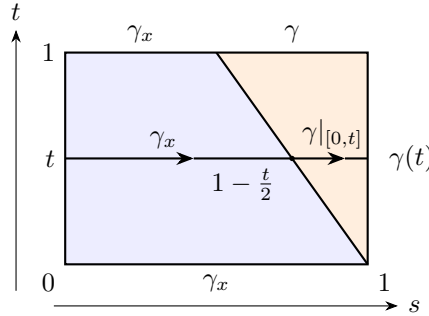


Figure 17: The path β_t runs along a horizontal slice. The joining point moves from $s = 1$ at $t = 0$ to $s = \frac{1}{2}$ at $t = 1$.

Lemma 7.10. *The lift of γ starting at x is*

$$\tilde{\gamma}(t) = (\gamma(t), [\beta_t]).$$

Proof. We check continuity using the local sheets. Fix $t_0 \in I$ and choose a simply connected open neighborhood U of $\gamma(t_0)$. For t in a sufficiently small interval about t_0 , the segment of γ between t_0 and t lies in U . The class $[\beta_t]$ is obtained from $[\beta_{t_0}]$ by extending along this segment, with the reverse direction when $t < t_0$. Reparametrizing the pieces and cancelling any part traversed forwards and backwards gives this equality of path classes. Thus, in the local product description over U , the map has the form

$$t \mapsto (\gamma(t), [\beta_{t_0}]),$$

which is continuous. It projects to γ and starts at x , so uniqueness of path lifting proves the claim. $\square$

Taking $x = \tilde{y}_0$, every point $(y, [\gamma])$ of $\tilde{Y}$ can be reached by such a lifted path. At $t = 1$ the initial constant piece can be removed by a reparametrization, leaving the class $[\gamma]$. Consequently, $\tilde{Y}$ is path-connected.

The fibre over y_0 is

$$S = f^{-1}(y_0) = P(y_0, y_0) = \pi_1(Y, y_0) = G,$$

where we suppress the fixed first coordinate y_0 . We can now compute its monodromy.

Lemma 7.11. *Under this identification of the fibre with G ,*

$$a_\gamma(x) = x[\gamma], \quad \text{act}(g)(x) = xg^{-1}.$$

The bijection $x \mapsto x^{-1}$ identifies the covariant monodromy action with the left translation action of G on itself.

Proof. If γ is a loop, then at $t = 1$ the formula for β_t is precisely the loop $\gamma_x\gamma$. Hence its class is $x[\gamma]$, proving the formula for a_γ . Passing to $a'_\gamma = a_{\gamma^{-1}}$ gives the formula for act . Finally,

$$(\text{act}(g)(x))^{-1} = (xg^{-1})^{-1} = gx^{-1},$$

which proves equivariance of the inversion bijection with respect to left translations. $\square$

Theorem 7.12. *A path-connected space with a basis of simply connected open neighborhoods has a universal covering. It is given by the path-class construction above.*

Proof. We have constructed a covering $f : \tilde{Y} \rightarrow Y$ with $\tilde{Y}$ path-connected. The stabilizer of the identity element of its fibre is trivial, since

$$\text{act}(g)(e) = g^{-1} = e \iff g = e.$$

By Proposition 6.13, $\pi_1(\tilde{Y}, \tilde{y}_0)$ is therefore trivial. Together with path-connectedness, this proves that $\tilde{Y}$ is simply connected. $\square$

8 From group actions to coverings

Lecture 8, 28 October 2026

Throughout this lecture, Y is path-connected and has a basis of simply connected open neighborhoods. Fix $y_0 \in Y$ and put $G = \pi_1(Y, y_0)$. We write e for the neutral element of G . Write

$$\tilde{f} : \tilde{Y} \longrightarrow Y$$

for the universal covering constructed in Lecture 7. The points of $\tilde{Y}$ are pairs $(y, [\gamma])$, where γ is a path from y_0 to y , considered up to homotopy with fixed endpoints.

We already have the monodromy functor $\Phi : \text{Cov}(Y) \rightarrow G\text{-}\mathbf{Set}$. We now construct a functor in the other direction, using an action of G on the whole universal covering space.

8.1 The action on the universal cover

For paths $\lambda, \gamma : I \rightarrow Y$ with $\lambda(1) = \gamma(0)$, their concatenation is the path

$$(\lambda\gamma)(t) = \begin{cases} \lambda(2t), & 0 \leq t \leq \frac{1}{2}, \\ \gamma(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

This agrees with the product already defined for loops. For the action below, λ is a loop at y_0 and γ is a path from y_0 to y .

Proposition 8.1. *The formula*

$$[\lambda] \cdot (y, [\gamma]) = (y, [\lambda\gamma])$$

defines a continuous left G -action on $\tilde{Y}$ which preserves the projection $\tilde{f}$. On each fibre $\tilde{f}^{-1}(y) = P(y_0, y)$ the action is free and transitive.

Proof. Changing λ by a based homotopy gives a homotopy of the displayed path on the first half of I . Changing γ by a homotopy with fixed endpoints gives one on the second half. Thus the formula is independent of both choices of representatives. The identity and associativity laws follow by removing a constant initial piece and by moving the joining times, as for the loop product.

To see freeness, suppose $[\lambda\gamma] = [\gamma]$. Follow both paths by the reverse path γ^{-1} . Contracting the excursion along γ and back gives a based homotopy from λ to the constant loop. Hence $[\lambda] = e$.

For transitivity, let $[\gamma], [\delta] \in P(y_0, y)$ and put $\lambda = \delta\gamma^{-1}$. In $\lambda\gamma$, the excursion backwards along γ and then forwards cancels. Thus $[\lambda] \cdot [\gamma] = [\delta]$.

Finally, let U be a simply connected open neighborhood of y . In the local description from equation (7.2),

$$\tilde{f}^{-1}(U) \cong U \times P(y_0, y), \quad g \cdot (z, c) = (z, g \cdot c).$$

Indeed, adding a loop at the beginning commutes, up to reparametrization, with extending a path at its end inside U . The action leaves the U -coordinate fixed and permutes the discrete set of sheets. It is therefore continuous. $\square$

On the fibre $\tilde{f}^{-1}(y_0) = G$, this action is left multiplication: $g \cdot x = gx$. The covariant monodromy action computed in Lemma 7.11 is $\text{act}(g)(x) = xg^{-1}$. We will use both formulas below.

8.2 Properly discontinuous actions

A continuous action of a group on a space means here an action by homeomorphisms.

Definition 8.2. A continuous action of a group G on a space X is *properly discontinuous* if every $x \in X$ has an open neighborhood V such that

$$gV \cap V = \emptyset \quad \text{for every } g \in G, \quad g \neq e.$$

The translates gV are then pairwise disjoint: an intersection $g_1V \cap g_2V \neq \emptyset$ would give $V \cap g_1^{-1}g_2V \neq \emptyset$, and hence $g_1 = g_2$.

Proposition 8.3. *The action of G on $\tilde{Y}$ is properly discontinuous.*

Proof. In a chart $\tilde{f}^{-1}(U) \cong U \times P(y_0, y)$, take the sheet $V = U \times \{c\}$ containing the chosen point. Its translate is $gV = U \times \{g \cdot c\}$. Freeness of the action on $P(y_0, y)$ gives $g \cdot c \neq c$ whenever $g \neq e$, so gV and V are disjoint. $\square$

For a group acting on a space X , write X/G for the set of orbits, with the *quotient topology*. If $q : X \rightarrow X/G$ is the orbit map, a subset $W \subseteq X/G$ is open precisely when $q^{-1}(W)$ is open in X . Equivalently, the open subsets are the images of open G -invariant subsets of X .

Proposition 8.4. *If G acts properly discontinuously on X , then the quotient map $q : X \rightarrow X/G$ is a covering.*

Proof. Choose $x \in X$ and an open neighborhood V as in the definition. The set

$$W = \bigcup_{g \in G} gV = q^{-1}(q(V))$$

is open and G -invariant, so $q(V)$ is open in X/G . The union is disjoint, and every orbit in $q(V)$ meets each gV exactly once. Thus each restriction $q|_{gV} : gV \rightarrow q(V)$ is a continuous bijection.

Fix g and let $r : q(V) \rightarrow gV$ be the inverse of this restriction. We check that r is continuous. If O is open in gV , then it is open in X , since gV is open. Moreover,

$$q^{-1}(q(O)) = \bigcup_{k \in G} kO$$

is open in X . By the definition of the quotient topology, $q(O)$ is open in X/G , and hence in $q(V)$. Since $r^{-1}(O) = q(O)$, this proves continuity of r . Therefore $q|_{gV}$ is a homeomorphism, and $q(V)$ is evenly covered. $\square$

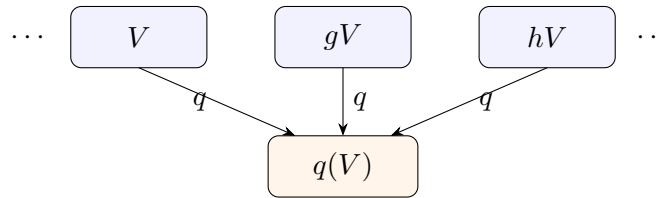


Figure 18: The distinct translates of V are disjoint open sheets over $q(V)$; each displayed arrow is a homeomorphism.

8.3 Quotients by subgroups

Let $\Gamma \subseteq G$ be a subgroup. Its action on $\tilde{Y}$ is still properly discontinuous. Therefore

$$\tilde{Y} \longrightarrow \tilde{Y}/\Gamma$$

is a covering, and it is a universal covering because $\tilde{Y}$ is simply connected.

Taking $\Gamma = G$ recovers the original base:

$$\tilde{Y}/G \cong Y.$$

Indeed, the action is transitive on each fibre of $\tilde{f}$, so $\tilde{f}$ identifies its fibres with the G -orbits. Locally,

$$(U \times P(y_0, y))/G \cong U,$$

because the action is trivial on U and transitive on $P(y_0, y)$. These local identifications show that the induced bijection $\tilde{Y}/G \rightarrow Y$ is a homeomorphism.

Proposition 8.5. *For every subgroup $\Gamma \subseteq G$, the induced map*

$$f_\Gamma : \tilde{Y}/\Gamma \longrightarrow Y$$

is a covering. Its monodromy G -set is isomorphic to G/Γ , with G acting by left multiplication.

Proof. For a simply connected open neighborhood U of y , there is a homeomorphism

$$f_\Gamma^{-1}(U) \cong U \times (P(y_0, y)/\Gamma),$$

which takes f_Γ to projection onto U . Since the second factor is discrete, U is evenly covered.

A point of the fibre over y_0 is represented by an element $g \in G$. Two elements g, g' represent the same point exactly when $g' = \gamma g$ for some $\gamma \in \Gamma$. This is equivalent to $g'^{-1}\Gamma = g^{-1}\Gamma$. Thus sending the point represented by g to $g^{-1}\Gamma$ gives a bijection from the fibre to G/Γ .

A loop with class $k \in G$ lifts from g to gk in $\tilde{Y}$. Hence covariant monodromy sends the point represented by g to the point represented by gk^{-1} . Under the bijection above, this becomes

$$g^{-1}\Gamma \longmapsto (gk^{-1})^{-1}\Gamma = kg^{-1}\Gamma,$$

which is the left action of k on G/Γ . □

To construct the inverse functor for all G -sets, we use a construction which accepts an arbitrary action directly.

8.4 A covering associated to a G -set

Let S be a left G -set. Take one copy of $\tilde{Y}$ for each $s \in S$, and denote their disjoint union by $\tilde{Y} \times S$. There is a diagonal left action:

$$g \cdot (\tilde{y}, s) = (g \cdot \tilde{y}, g \cdot s).$$

It is properly discontinuous. In fact, if V is an open neighborhood in $\tilde{Y}$ with disjoint translates, then $V \times \{s\}$ has disjoint translates as well. Thus the quotient map from $\tilde{Y} \times S$ is a covering. Our main interest is the following map to Y .

Definition 8.6. Set

$$Y_S = (\tilde{Y} \times S)/G, \quad f_S : Y_S \longrightarrow Y, \quad f_S([(\tilde{y}, s)]) = \tilde{f}(\tilde{y}).$$

Here $[(\tilde{y}, s)]$ denotes the orbit of $(\tilde{y}, s)$ under the diagonal action.

The map f_S is well-defined because $\tilde{f}$ is unchanged by the G -action, and it is continuous by the quotient topology.

Proposition 8.7. *The map $f_S : Y_S \rightarrow Y$ is a covering. For every simply connected open neighborhood U of y , there is a homeomorphism*

$$\varphi_{U,y} : f_S^{-1}(U) \longrightarrow U \times ((P(y_0, y) \times S)/G) \quad (8.1)$$

such that

$$\text{pr}_1 \circ \varphi_{U,y} = f_S|_{f_S^{-1}(U)}.$$

Here the space on the right of (8.1) is the disjoint union of copies of U indexed by the set $(P(y_0, y) \times S)/G$.

Proof. The local product description of $\tilde{Y}$ is G -equivariant, with G acting only on $P(y_0, y)$. Thus $f_S^{-1}(U)$ is the quotient of copies of U labelled by pairs $(c, s) \in P(y_0, y) \times S$. The action leaves the U -coordinate fixed, so

$$\varphi_{U,y}([(z, c, s)]) = (z, [(c, s)])$$

is a well-defined bijection.

Fix an orbit L of pairs (c, s) . The copies of U labelled by pairs in L form an open, G -invariant subset before taking the quotient. Their image V_L is therefore open in $f_S^{-1}(U)$. The restriction $f_S|_{V_L} : V_L \rightarrow U$ is a continuous bijection. Choosing one representative $(c, s) \in L$, its inverse is the continuous map

$$U \longrightarrow V_L, \quad z \longmapsto [(z, c, s)],$$

obtained by including one copy of U and then applying the quotient map. Hence $f_S|_{V_L}$ is a homeomorphism.

The sets V_L are disjoint and cover $f_S^{-1}(U)$. Consequently $\varphi_{U,y}$ is a homeomorphism with the stated projection identity, and U is evenly covered. $\square$

8.5 The functor on morphisms

We have assigned a covering to each G -set. We must also assign a morphism of coverings to each equivariant map, and verify preservation of identities and composition.

Proposition 8.8. *The assignments*

$$\begin{aligned} S &\longmapsto (f_S : Y_S \rightarrow Y), \\ (\beta : S_1 \rightarrow S_2) &\longmapsto (\Psi(\beta) : Y_{S_1} \rightarrow Y_{S_2}), \end{aligned}$$

where

$$\Psi(\beta)([(\tilde{y}, s)]) = [(\tilde{y}, \beta(s))],$$

define a functor

$$\Psi : G\text{-Set} \longrightarrow \text{Cov}(Y).$$

Proof. The object assignment gives a covering by Proposition 8.7. For the morphism assignment, equivariance of β gives

$$(g \cdot \tilde{y}, \beta(g \cdot s)) = (g \cdot \tilde{y}, g \cdot \beta(s)).$$

Thus changing an orbit representative does not change the image orbit, and $\Psi(\beta)$ is well-defined.

The map $\text{id}_{\tilde{Y}} \times \beta$ sends the copy of $\tilde{Y}$ labelled by s identically to the copy labelled by $\beta(s)$. It is therefore continuous on each open copy of $\tilde{Y}$, hence continuous on their union. It induces a continuous map on the quotients. In the local product descriptions, this map is

$$(z, [(c, s)]) \longmapsto (z, [(c, \beta(s))]).$$

It leaves z unchanged, so $f_{S_2} \circ \Psi(\beta) = f_{S_1}$. Hence it is a morphism in $\text{Cov}(Y)$.

Finally, the formula gives

$$\Psi(\text{id}_S) = \text{id}_{Y_S}.$$

For equivariant maps $\beta : S_1 \rightarrow S_2$ and $\delta : S_2 \rightarrow S_3$, both $\Psi(\delta \circ \beta)$ and $\Psi(\delta) \circ \Psi(\beta)$ send $[(\tilde{y}, s)]$ to $[(\tilde{y}, \delta(\beta(s)))]$. Therefore

$$\Psi(\delta \circ \beta) = \Psi(\delta) \circ \Psi(\beta).$$

□

8.6 Recovering the original G -set

We now have functors in both directions. Start with a G -set S , construct Y_S , and take the fibre over y_0 . Since $\tilde{f}^{-1}(y_0) = G$, the resulting set is

$$(\Phi \circ \Psi)(S) = (G \times S)/G,$$

where the quotient uses the diagonal action $k \cdot (g, s) = (kg, k \cdot s)$. This set need not be literally equal to S ; we must construct an isomorphism of G -sets and check that it is natural in S .

Proposition 8.9. *The maps*

$$\theta_S : (\Phi \circ \Psi)(S) \longrightarrow S, \quad \theta_S([(g, s)]) = g^{-1} \cdot s,$$

form a natural isomorphism $\theta : \Phi \circ \Psi \Rightarrow \text{id}_{G\text{-Set}}$.

Proof. The underlying bijection. The formula is constant on diagonal orbits, since

$$(kg)^{-1} \cdot (k \cdot s) = g^{-1} \cdot s.$$

Every orbit has exactly one representative of the form (e, t) : applying g^{-1} to (g, s) gives $(e, g^{-1} \cdot s)$. Thus θ_S is a bijection, with inverse $s \mapsto [(e, s)]$.

The monodromy action. Let λ be a based loop with $[\lambda] = k$. Starting at $[(g, s)]$, lift λ to $\tilde{Y}$ starting at g , keep s constant, and project the resulting path to Y_S . By Lemma 7.11, the lifted path in $\tilde{Y}$ ends at gk . Consequently,

$$a_\lambda([(g, s)]) = [(gk, s)], \quad \text{act}(k)([(g, s)]) = [(gk^{-1}, s)].$$

The second formula uses our convention $\text{act}([\lambda]) = a_{\lambda^{-1}}$. It follows that

$$\theta_S(\text{act}(k)([(g, s)])) = (gk^{-1})^{-1} \cdot s = k \cdot (g^{-1} \cdot s) = k \cdot \theta_S([(g, s)]).$$

Thus θ_S is equivariant, so it is an isomorphism in $G\text{-Set}$.

Naturality. For an equivariant map $\beta : S_1 \rightarrow S_2$, the induced map on fibres sends $[(g, s)]$ to $[(g, \beta(s))]$. Therefore

$$\theta_{S_2}([(g, \beta(s))]) = g^{-1} \cdot \beta(s) = \beta(g^{-1} \cdot s) = \beta(\theta_{S_1}([(g, s)])).$$

This is exactly the commutativity of the naturality square:

$$\begin{array}{ccc} (G \times S_1)/G & \xrightarrow{\theta_{S_1}} & S_1 \\ (\Phi \circ \Psi)(\beta) \downarrow & & \downarrow \beta \\ (G \times S_2)/G & \xrightarrow{\theta_{S_2}} & S_2 \end{array}$$

Hence the isomorphisms θ_S form a natural isomorphism.

□

8.7 The comparison map for a covering

For the other composite, start with a covering $f : X \rightarrow Y$ and its monodromy G -set $S = f^{-1}(y_0)$. Applying Ψ gives

$$Y_S = (\tilde{Y} \times S)/G \longrightarrow Y.$$

We seek an isomorphism of coverings $Y_S \rightarrow X$, natural in f . Since Y_S is a quotient, first construct a map

$$F : \tilde{Y} \times S \longrightarrow X.$$

For each $s \in S$, define

$$F_s : \tilde{Y} \longrightarrow X, \quad F_s(y, [\gamma]) = \tilde{\gamma}_s(1),$$

where $\tilde{\gamma}_s$ is the lift of γ to X starting at s . Thus F_s takes a path class to the endpoint of its lift. Set $F(\tilde{y}, s) = F_s(\tilde{y})$.

Lemma 8.10. *For every $s \in S$, the map F_s is well-defined and is a local homeomorphism. Moreover, $f \circ F_s = \tilde{f}$.*

Proof. If two paths represent the same element of $P(y_0, y)$, lift a homotopy between them while keeping the initial point equal to s . The lifted endpoints form a continuous map from I to the discrete fibre $f^{-1}(y)$, so they are constant. This is the endpoint argument from Proposition 6.3, now with possibly different initial and final points in Y . It proves that F_s is well-defined.

Choose a simply connected open neighborhood U of y which is evenly covered by f . Use the identifications

$$\tilde{f}^{-1}(U) \cong U \times P(y_0, y), \quad f^{-1}(U) \cong U \times f^{-1}(y).$$

Fix $[\gamma] \in P(y_0, y)$ and $z \in U$. Choose a path η inside U from y to z , as in Figure 16. Under the first identification, the coordinate $(z, [\gamma])$ represents the point $(z, [\gamma\eta])$ of $\tilde{Y}$.

Put $x = \tilde{\gamma}_s(1)$, and let V be the sheet of $f^{-1}(U)$ containing x . The lift of $\gamma\eta$ starting at s first follows $\tilde{\gamma}_s$ and then lifts η starting at x . Since $f|_V : V \rightarrow U$ is a homeomorphism, this second lift is $(f|_V)^{-1} \circ \eta$. Its endpoint is $(f|_V)^{-1}(z)$, which has coordinates (z, x) under the second identification.

Consequently, in these coordinates, F_s is given by

$$(z, [\gamma]) \longmapsto (z, \tilde{\gamma}_s(1)).$$

For each fixed $[\gamma]$, this is the identity $U \rightarrow U$ between the corresponding copies of U . It follows that F_s is a local homeomorphism and $f \circ F_s = \tilde{f}$. $\square$

The maps F_s together give a continuous map $F : \tilde{Y} \times S \rightarrow X$, since each is continuous on the corresponding open copy of $\tilde{Y}$. The remaining task is to show that F descends to a homeomorphism $Y_S \rightarrow X$ and that these homeomorphisms are natural in the covering $f : X \rightarrow Y$. We will complete these verifications in the next lecture.

9 Classification of coverings and a cubic example

Lecture 9, 29 October 2026

We finish the comparison between a covering and the covering reconstructed from its monodromy action. We then use this equivalence to study automorphisms of coverings and work out the example $x \mapsto x^3 - x$.

9.1 Recovering a covering from its fibre

As before, let Y be path-connected with a basis of simply connected open neighborhoods, fix $y_0 \in Y$, and put $G = \pi_1(Y, y_0)$. Write $\tilde{f} : \tilde{Y} \rightarrow Y$ for the universal covering. For a covering $f : X \rightarrow Y$, set $S = f^{-1}(y_0)$, with its covariant monodromy action.

Recall the map constructed in Section 8.7:

$$F : \tilde{Y} \times S \longrightarrow X, \quad F((y, [\gamma]), s) = \tilde{\gamma}_s(1).$$

Here $\tilde{\gamma}_s$ is the lift of γ starting at s . By Lemma 8.10, F is well-defined and is a local homeomorphism. The space $\tilde{Y} \times S$ is the disjoint union of its open copies of $\tilde{Y}$.

Proposition 9.1. *The map F induces an isomorphism of coverings*

$$\overline{F} : Y_S = (\tilde{Y} \times S)/G \longrightarrow X, \quad \overline{F}([(y, s)]) = F(y, s).$$

In particular, $f \circ \overline{F} = f_S$.

Proof. Compatibility with the action. We must show that

$$F(g \cdot \tilde{y}, g \cdot s) = F(\tilde{y}, s).$$

Write $g = [\lambda]$ and $\tilde{y} = (y, [\gamma])$. Then $g \cdot \tilde{y} = (y, [\lambda\gamma])$. Put $s' = g \cdot s$. By our convention, s' is the endpoint of the lift of λ^{-1} starting at s . Reversing that lift gives a path δ from s' to s which projects to λ .

The concatenation $\delta\tilde{\gamma}_s$ is therefore the lift of $\lambda\gamma$ starting at s' . Its endpoint is still $\tilde{\gamma}_s(1)$. Consequently,

$$F(g \cdot \tilde{y}, g \cdot s) = \tilde{\gamma}_s(1) = F(\tilde{y}, s).$$

Thus F is constant on the diagonal orbits and induces a continuous map $\overline{F}$ on the quotient.

The identity $f \circ \overline{F} = f_S$ follows from the definition of F .

Local homeomorphism. For a sufficiently small simply connected open neighborhood U of $y \in Y$, we have the product descriptions

$$\begin{aligned} \tilde{f}^{-1}(U) \times S &\cong U \times (P(y_0, y) \times S), \\ f_S^{-1}(U) &\cong U \times ((P(y_0, y) \times S)/G), \\ f^{-1}(U) &\cong U \times f^{-1}(y). \end{aligned}$$

As in Lemma 8.10, F sends each copy of U identically to the copy indexed by the endpoint of the lifted path. The action of G changes only the indices of these copies, so the same is true of $\overline{F}$ after taking the quotient. Thus each copy of U in Y_S is mapped isomorphically to a copy of U in X , and $\overline{F}$ is a local homeomorphism.

Surjectivity. As in the proof of Theorem 6.15, every point $x \in X$ can be joined to a point of S by a path in X : lift a path from $f(x)$ to y_0 and reverse the lift. Choose such a path δ with $\delta(0) = s \in S$ and $\delta(1) = x$, and set $\gamma = f \circ \delta$. By uniqueness of path lifting, $\delta = \tilde{\gamma}_s$. Therefore

$$\overline{F}([(f(x), [\gamma]), s)]) = x.$$

Injectivity. Suppose

$$F(\tilde{y}_1, s_1) = F(\tilde{y}_2, s_2) = x, \quad \tilde{y}_i = (y_i, [\gamma_i]).$$

Applying f gives $y_1 = y_2 = f(x)$; denote this point by y . The lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$, starting at s_1, s_2 , both end at x , as in Figure 19. Set

$$\lambda = \gamma_1 \gamma_2^{-1}, \quad g = [\lambda] \in G.$$

The path $\tilde{\gamma}_2 \tilde{\gamma}_1^{-1}$ lifts λ^{-1} from s_2 to s_1 . Hence $g \cdot s_2 = s_1$. Also,

$$g \cdot \tilde{y}_2 = (y, [\gamma_1 \gamma_2^{-1} \gamma_2]) = (y, [\gamma_1]) = \tilde{y}_1.$$

Thus the two pairs belong to the same diagonal orbit. Their images in Y_S coincide, proving injectivity.

A bijective local homeomorphism is a homeomorphism: its inverse is continuous on each of the open sets supplied by the local homeomorphisms. This proves the proposition. $\square$

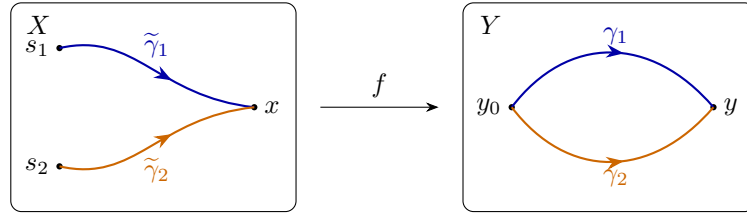


Figure 19: The lifts have the same endpoint. Following $\tilde{\gamma}_1$ and then $\tilde{\gamma}_2^{-1}$ joins s_1 to s_2 and projects to $\gamma_1 \gamma_2^{-1}$.

9.2 Naturality and the equivalence of categories

The homeomorphism just constructed depends on the covering f . Write it as $\bar{F}_f : Y_{S_f} \rightarrow X$, where $S_f = f^{-1}(y_0)$.

Proposition 9.2. *The maps $\bar{F}_f$ form a natural isomorphism*

$$\Psi \circ \Phi \implies \text{id}_{\text{Cov}(Y)}.$$

Proof. Let $\alpha : X_1 \rightarrow X_2$ be a morphism of coverings, so $f_2 \circ \alpha = f_1$. Its restriction to the fibres is $\alpha_0 : S_1 \rightarrow S_2$. We must prove that the square

$$\begin{array}{ccc} Y_{S_1} & \xrightarrow{\bar{F}_{f_1}} & X_1 \\ \Psi(\alpha_0) \downarrow & & \downarrow \alpha \\ Y_{S_2} & \xrightarrow{\bar{F}_{f_2}} & X_2 \end{array}$$

commutes. Take a path γ from y_0 to y and $s \in S_1$. If $\tilde{\gamma}_s$ is its lift to X_1 , then $\alpha \circ \tilde{\gamma}_s$ is a lift of γ to X_2 starting at $\alpha(s)$. By uniqueness of lifts, it is the lift used to define $\bar{F}_{f_2}$. Evaluating at the endpoint gives

$$\alpha(\bar{F}_{f_1}([(y, [\gamma]), s]))) = \bar{F}_{f_2}([(y, [\gamma]), \alpha_0(s)]).$$

The right side is $\bar{F}_{f_2} \circ \Psi(\alpha_0)$ evaluated at the same point. This proves naturality. $\square$

Proposition 8.9 already gave a natural isomorphism $\Phi \circ \Psi \implies \text{id}_{G\text{-Set}}$. We have now constructed both natural isomorphisms, completing the proof of Theorem 7.7:

$$\text{Cov}(Y) \simeq G\text{-Set}.$$

9.3 One-object categories

A category with one object is the same data as a monoid: its elements are the endomorphisms of that object, multiplication is composition, and the neutral element is the identity morphism. A functor between such categories is a homomorphism of monoids.

Exercise 9.3. Let M, N be monoids, regarded as categories with one object. Describe natural transformations and natural isomorphisms between functors $M \rightarrow N$. Show that these categories are equivalent if and only if the monoids M, N are isomorphic.

If $\varphi : M \rightarrow N$ and $\psi : N \rightarrow M$ exhibit an equivalence, their composites need not be identity maps. Show how the natural isomorphisms give invertible elements $\theta \in M$ and $\eta \in N$ such that

$$(\psi \circ \varphi)(m) = \theta^{-1}m\theta, \quad (\varphi \circ \psi)(n) = \eta^{-1}n\eta.$$

9.4 Automorphisms of coverings

Definition 9.4. An *automorphism* of a covering $f : X \rightarrow Y$, also called a *deck transformation*, is a homeomorphism $\alpha : X \rightarrow X$ satisfying $f \circ \alpha = f$. These maps form a group under composition, denoted $\text{Aut}(X, f)$.

An automorphism of a G -set S is an equivariant bijection $u : S \rightarrow S$. Thus u is a bijection satisfying

$$u(g \cdot s) = g \cdot u(s) \quad (g \in G, s \in S).$$

Write $\text{Aut}_G(S)$ for this group.

We use the following general fact about equivalences.

Lemma 9.5. Let $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ be an equivalence of categories. For any objects $X_1, X_2 \in \mathcal{C}$, the map

$$\text{Hom}_{\mathcal{C}}(X_1, X_2) \longrightarrow \text{Hom}_{\mathcal{D}}(\Phi(X_1), \Phi(X_2)), \quad \alpha \longmapsto \Phi(\alpha),$$

is a bijection. In particular, it induces a group isomorphism

$$\text{Aut}_{\mathcal{C}}(X) \cong \text{Aut}_{\mathcal{D}}(\Phi(X)).$$

Proof. Choose $\Psi : \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms $\theta : \Psi\Phi \Rightarrow \text{id}_{\mathcal{C}}$ and $\eta : \Phi\Psi \Rightarrow \text{id}_{\mathcal{D}}$. Naturality of θ gives, for every $\alpha : X_1 \rightarrow X_2$,

$$\alpha = \theta_{X_2} \circ \Psi\Phi(\alpha) \circ \theta_{X_1}^{-1}.$$

Thus $\Phi(\alpha)$ determines α , proving injectivity. The same argument with η shows that Ψ is injective on each set of morphisms as well.

For surjectivity, let $\beta : \Phi(X_1) \rightarrow \Phi(X_2)$ and put

$$\alpha = \theta_{X_2} \circ \Psi(\beta) \circ \theta_{X_1}^{-1}.$$

Applying the naturality identity to this α gives

$$\Psi\Phi(\alpha) = \theta_{X_2}^{-1} \circ \alpha \circ \theta_{X_1} = \Psi(\beta).$$

Injectivity for Ψ now implies $\Phi(\alpha) = \beta$.

For $X_1 = X_2 = X$, the bijection preserves composition and the identity, so it identifies the invertible endomorphisms and their groups. $\square$

Applying the lemma to the monodromy equivalence gives the group isomorphism

$$\text{Aut}(X, f) \longrightarrow \text{Aut}_G(S), \quad \alpha \longmapsto \alpha|_S.$$

9.5 From a cubic polynomial to a covering

Consider the polynomial map

$$P : \mathbb{C} \longrightarrow \mathbb{C}, \quad P(x) = x^3 + ax + b.$$

Identify $\mathbb{C}$ with the sphere minus one point by stereographic projection. Since $|P(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$, the map extends continuously to

$$\hat{P} : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}, \quad \hat{P}(\infty) = \infty.$$

This map of spheres is not a covering. The fibre over ∞ consists of one point, whereas a typical finite fibre consists of three points.

Assume $a \neq 0$, and choose c with $c^2 = -a/3$. The derivative vanishes at c and $-c$, and their images are

$$P(c) = b - 2c^3, \quad P(-c) = b + 2c^3.$$

These are two distinct *critical values*. The factorizations

$$P(x) - P(c) = (x - c)^2(x + 2c), \quad P(x) - P(-c) = (x + c)^2(x - 2c)$$

show that each has exactly two distinct preimages. Every other finite value has three distinct preimages. If $a = 0$, there is instead just one finite critical value, namely b .

We now specialize to

$$f(x) = x^3 - x.$$

Put $r = 1/\sqrt{3}$. The critical points are $-r, r$, and we label the critical values in increasing order:

$$y_1 = f(r) = -\frac{2}{3\sqrt{3}}, \quad y_2 = f(-r) = \frac{2}{3\sqrt{3}}.$$

Their complete fibres are

$$f^{-1}(y_1) = \{-2r, r\}, \quad f^{-1}(y_2) = \{-r, 2r\}.$$

Thus removing the critical values and all their preimages gives

$$Y = \mathbb{C} \setminus \{y_1, y_2\}, \quad X = \mathbb{C} \setminus \{-2r, -r, r, 2r\}.$$

The four removed points, in increasing order on the real axis, are $x_1 = -2r$, $x_2 = -r$, $x_3 = r$, and $x_4 = 2r$. The two outer points are removed as well because they map to the excluded values, even though f' does not vanish there.

Proposition 9.6. *The restriction $f : X \rightarrow Y$ is a three-sheeted covering.*

Proof. For every $y \in Y$, the polynomial $x^3 - x - y$ has three distinct roots s_1, s_2, s_3 , and $f'(s_i) \neq 0$. The inverse function theorem gives pairwise disjoint open neighborhoods of these roots on which f is a homeomorphism onto an open neighborhood of y .

Choose a small open disk U about y , contained in Y and in all three image neighborhoods. Restricting the three inverse branches to U gives disjoint open sets V_1, V_2, V_3 , each mapped homeomorphically onto U . For every $z \in U$, these branches already supply three distinct roots of $x^3 - x - z$. A cubic has no further roots, so

$$f^{-1}(U) = V_1 \sqcup V_2 \sqcup V_3.$$

Thus U is evenly covered. □

For real y strictly between y_1 and y_2 , all three roots are real. Outside this interval, there is one real root and two nonreal complex conjugate roots. The complex covering still has three points in every fibre.

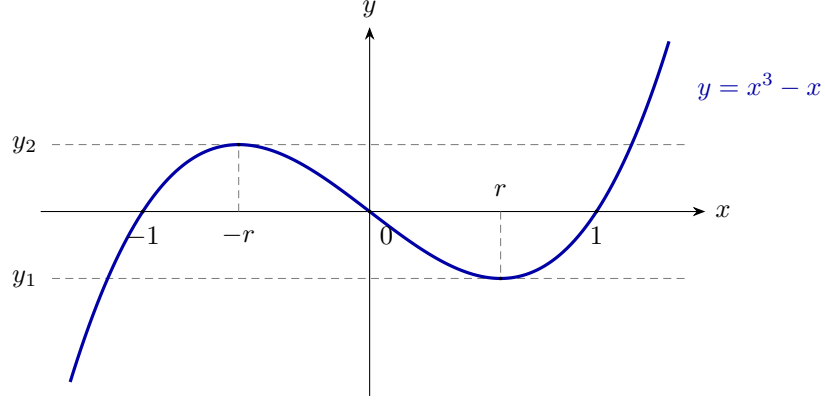


Figure 20: The real graph of the cubic. Two real roots coalesce at each of the critical values y_1, y_2 .

9.6 Monodromy and deck transformations of the cubic

Choose the basepoint $y_0 = 0$. Its fibre is

$$S = f^{-1}(0) = \{-1, 0, 1\}.$$

Let $g_1, g_2 \in G = \pi_1(Y, 0)$ be represented by the counterclockwise loops around y_1, y_2 drawn in Figure 21. Each approaches its puncture from 0, makes one turn, and returns.

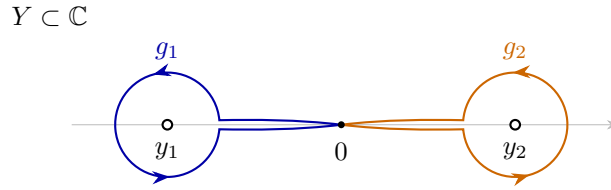


Figure 21: The two based loops used to compute monodromy. Both turn counterclockwise around their respective punctures.

Proposition 9.7. *For the monodromy action on $S = \{-1, 0, 1\}$, act_{g_1} exchanges 0 and 1 and fixes -1 , while act_{g_2} exchanges -1 and 0 and fixes 1. The image of the action is the full permutation group $\mathfrak{S}_3$.*

Proof. We lift the loop representing g_1 starting at $x = 0$. Along the real segment from $y = 0$ towards $y_1 = f(r)$, the lift moves along the real axis from 0 towards r ; see Figure 20. Stop when y is close to y_1 . Next, y goes once counterclockwise around a small circle centered at y_1 .

To continue this lift around the circle, use

$$f(x) - y_1 = (x - r)^2(x + 2r).$$

Choose a square root of $x + 2r$ near r , positive at r , and put

$$v = (x - r)\sqrt{x + 2r}.$$

This defines a change of coordinates in a neighborhood of $x = r$. In the new coordinate, we continue the lift by choosing $v(t)$ continuously so that

$$v(t)^2 = y(t) - y_1.$$

For our lift, v starts negative, since $x < r$. As $y(t) - y_1$ makes one turn around 0, its square root $v(t)$ makes half a turn and ends positive. The corresponding point x now lies to the right of r . The return along the real segment therefore lifts to a path ending at $x = 1$. Thus $a_{g_1}(0) = 1$.

The same argument starting at 1 gives $a_{g_1}(1) = 0$. Since a_{g_1} is a permutation of S , it fixes the remaining point -1 . The case of g_2 is analogous: a_{g_2} exchanges -1 and 0 and fixes 1 .

Our covariant action uses the inverse endpoint permutation. Since each of these transpositions is its own inverse, $\text{act}_{g_i} = a_{g_i}$ here. The two transpositions generate $\mathfrak{S}_3$, proving surjectivity of

$$\text{act} : \pi_1(Y, 0) \longrightarrow \mathfrak{S}_3.$$

□

The identification of the fundamental group of the two-punctured plane with the free group on two generators was announced in Section 4.7. With this identification, the example gives the surjective homomorphism

$$F_2 = \langle g_1, g_2 \rangle \longrightarrow \mathfrak{S}_3$$

specified by the two transpositions above.

Proposition 9.8. *The covering $f : X \rightarrow Y$ has no deck transformation other than the identity:*

$$\text{Aut}(X, f) = \{\text{id}_X\}.$$

Proof. By Lemma 9.5 applied to the monodromy equivalence, we must compute $\text{Aut}_G(S)$. Let $u : S \rightarrow S$ be an equivariant bijection. An equivariant map sends points fixed by a group element to points fixed by that same element, since

$$g \cdot s = s \implies g \cdot u(s) = u(g \cdot s) = u(s).$$

The unique point fixed by g_1 is -1 , so $u(-1) = -1$. The unique point fixed by g_2 is 1 , so $u(1) = 1$. Bijectivity then forces $u(0) = 0$. Thus u is the identity, and so is the corresponding deck transformation. □

In this example the monodromy image is $\mathfrak{S}_3$, while the group of automorphisms of the covering is trivial.